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ABSTRACT

The inspiration for this text was provided by an exposure to the flow proof approach to a proof format as opposed to the conventional two-column approach. Historical background is included, to provide a frame of reference to give the student an appreciation of the subject. The basic constructions are introduced early and briefly, to aid the process of induction, encourage the student to experiment, and provide an approach to proof. Transformations and similarity are integrated, with congruence reapproached in the unit on transformations through isometries so that all three topics are tied together. Following an introduction, the 22 chapters concern: history of geometry; basic constructions; induction; sets and Venn diagrams; logic and deductive reasoning; the real numbers and distance; the structure of geometry; flow proof; angles and angle measure; indirect proof; congruence; geometric inequalities; parallel lines; quadrilaterals; lines and planes in space; transformations; similarity; circles and spheres; construction problems and locus; plane coordinate geometry; areas of polygons and circles; and areas and volumes of space figures. (MNS)

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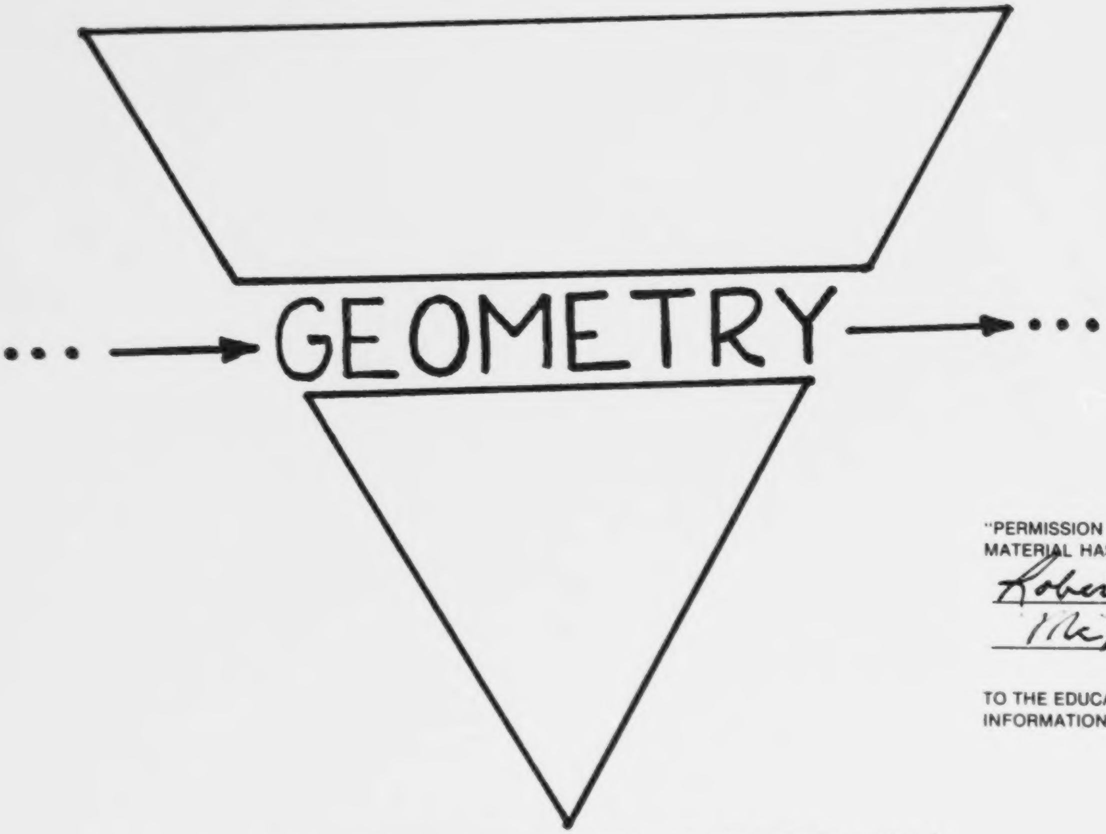

... → GEOMETRY → ...

A FLOW PROOF APPROACH

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ROBERT McMURRAY

SE 050 226

Robert McMurray
8129 San Francisco NE
Albuquerque, NM 87109

COMMENTS
Robert McMurray

The inspiration for this text was provided by an exposure to the flow proof approach to a proof format as opposed to the conventional two column approach. Although the text would not have been written without this inspiration, it also reflects some other ideas concerning the teaching of geometry which have been brewing in my mind over the years. I feel the content basically follows SMSG guidelines, but I would like to comment on a few topics before discussing the flow proof aspects of the text.

I have felt for many years that any math textbook should include some historical background pertinent to that course, particularly geometry. Even our best math students normally complete 3 or 4 years of high school math and still do not realize that there is such a thing as a history of mathematics. I like the vignettes normally included to spotlight significant figures of the past, but they do not tie the subject together. Geometry in particular requires a frame of reference to give the student an appreciation of the subject. I tried to write this unit so as to be enlightening without being overdone. I feel it can be both an educational experience and a motivating factor if written properly. It could be treated as a single lesson, but references may be made to it throughout the course.

The basic constructions are introduced early and briefly for two primary reasons. One is to aid the process of induction throughout the course and to encourage the student to experiment. The other reason is related to proof. I feel that certain theorems should be approached by requiring the student to construct the figure according to the conditions of the hypothesis. So many students actually draw a figure on the basis of the conclusion (e.g. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.) that they do not really distinguish the two in a verbal statement of a theorem. The requirement of constructing the conditions of the hypothesis may help to clarify that which is "given".

The role of geometric transformations in the sequence was the result of a search for an acceptable approach to the AA similarity theorem. This dictated a close integration of transformations and similarity which I felt turned out to be a happy event. It would have been logical to have approached congruence initially through transformations also, but I decided against it mainly because I wanted flow proof to be the dominant "new" feature of the text. However, congruence is reapproached in the unit on transformations through isometries so that all three topics are tied together.

Comments Related to Flow Proof

Considerable thought was given to two major aspects of utilizing the flow proof approach.

1. Laying the groundwork so that the organization of statements would seem completely natural to the student before the introduction of proof as an exercise.
2. Taking full advantage of the positive qualities possessed by flow proof and its relation to the deductive process in a manner that the student can grasp.

A major step toward achieving these ends is taken with the introduction of elementary symbolic logic. The emphasis is on the implication - the foundation stone of flow proof, the biconditional - the frequently neglected characteristic of definitions, and the negation - used mainly in indirect proof.

The Law of Detachment and the Chain Rule basically constitute the deductive process. Yet I have never seen a geometry text (there are many I haven't seen, of course) that clarifies the role that these principles play in a geometry proof. In the statement of the Law of Detachment,

If $p \rightarrow q$ is true
and p is true
then q is true

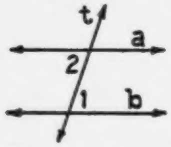
it should be stressed that $p \rightarrow q$ is a general statement and that p is detached for a specific case. Likewise, q is deduced for a specific case.

One currently used geometry text includes the following exercise for the student.

If George lives in Utica, then George lives in New York.
George lives in Utica.
State the conclusion.

The student may be happy with his conclusion, but will completely miss the point as far as any connection it may have with a proof in geometry.

I have included both nonmathematical and mathematical examples and exercises such as the following in non-proof situations.



If two parallel lines are cut by a transversal, then alternate interior angles are congruent.	$p \rightarrow q$ is true
$a \parallel b$ cut by transversal t	p is true
$\therefore \angle 1 \cong \angle 2$	$\therefore q$ is true

When the same conditions appear later in a proof, they would be written as follows.

$$\ldots \xrightarrow{④} a \parallel b \xrightarrow{⑤} \angle 1 \cong \angle 2 \xrightarrow{⑥} \ldots$$

⋮

④

⑤ If 2 ∥ lines are cut by a transversal, then alternate interior ∠s are ≅.

⑥

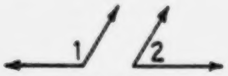
⋮

The Law of Detachment is once again stressed in the second instance and some exercises require the student to rearrange statements and reasons from a proof into the general form of the Law of Detachment. This is much easier to do and is more meaningful with a flow proof organization than with two column proof. So many students now taking geometry miss the fact that a reason must be related to two (or more) statements rather than just the one they are writing in the left column.

Another piece of groundwork is laid in the treatment of definitions. Some definitions are used as reasons in a proof and some are not. Those that are used as reasons are stated both verbally and symbolically as biconditionals.

Example: Two angles are supplementary if and only if the sum of their measures is 180.

$$\angle 1 \text{ suppl } \angle 2 \longleftrightarrow m\angle 1 + m\angle 2 = 180$$



It is stressed to the student that either specific implication may be made with the appropriate general statement whenever a definition is to be used.

Many exercises are included involving definitions, postulates, and the algebra properties in which the student sees the relationship of statements exactly as they will appear in a proof before he is exposed to proof itself.

One of the major advantages offered by flow proof is the opportunity to emphasize the Chain Rule in the deductive process. As each theorem is proved, it is stated symbolically following the conclusion so that the student sees it as it is now available for use in future proofs.

Example: If two angles are supplementary to the same angle, they are congruent to each other.

Given: $\angle 1$ suppl. $\angle 3$
$\angle 2$ suppl. $\angle 3$

Prove: $\angle 1 \cong \angle 2$

Argument

$$\ldots \longrightarrow \ldots \longrightarrow \ldots \longrightarrow \angle 1 \cong \angle 2$$

$$\therefore \left.\begin{array}{l} \angle 1 \text{ suppl. } \angle 3 \\ \angle 2 \text{ suppl. } \angle 3 \end{array}\right\} \longrightarrow \angle 1 \cong \angle 2$$

One of the major weaknesses of geometry students has been their failure to utilize that which they have already proved. If they constantly see the implication symbolically at the end of each proved theorem, it should encourage them to be more aware of its availability.

An experienced geometry teacher who has lived with the frustrations of presenting proof will not find flow proof to be a cure-all. But in this writer's opinion, it is superior to two column proof in every aspect of both teaching and learning. Teacher presentations are actually easier to prepare and to follow. It also provides for more flexibility in testing. Tests in a two column approach will frequently give statements and require the student to give reasons. This can be done as well with flow proof, but it is also possible to simply scramble the statements and require the student to organize them in a flow proof. When this is done in two column proof, the student can accidentally get it right. As the course progresses, less emphasis can be placed on writing reasons in the proof. If the student has the statements correctly organized in a flow proof, there is more assurance that he "sees" all the implications. This is more important than being able to write a precise verbal statement as a reason. He should, of course, be held responsible for knowing reasons, and in particular, to be able to recognize the Law of Detachment throughout the course.

Flow proof is probably more appropriate in geometry than in other areas of math, but most teachers already use the implication in many of their presentations. If the student were thoroughly familiar with flow proof, this use could be expanded considerably and would be much more meaningful.

Example:

$$\left.\begin{array}{l} \left.\begin{array}{l} \textcircled{1}\ \log_b M = m \xrightarrow{\textcircled{2}} M = b^m \\ \textcircled{3}\ \log_b N = n \xrightarrow{\textcircled{4}} N = b^n \end{array}\right\} \xrightarrow{\textcircled{5}} MN = b^m b^n \\ \qquad\qquad\qquad\qquad\qquad \textcircled{6}\ b^m b^n = b^{m+n} \end{array}\right\} \xrightarrow{\textcircled{7}} MN = b^{m+n} \xrightarrow{\textcircled{8}}$$

$$\xrightarrow{\textcircled{8}} \left.\begin{array}{l} \log_b(MN) = m+n \\ \textcircled{1}\ \log_b M = m \\ \textcircled{3}\ \log_b N = n \end{array}\right\} \xrightarrow{\textcircled{9}} \log_b(MN) = \log_b M + \log_b N$$

Most algebraic proofs and solutions of equations would consist of a horizontal string of equivalent statements using the biconditional. In the case of solving an equation with extraneous solutions, the non-equivalence could be emphasized by showing where the reversibility would not occur.

Example: Solve $\sqrt{11 - x} = 3 + \sqrt{x + 6}$

$$\textcircled{1}\ \sqrt{11 - x} = 3 + \sqrt{x + 6} \xrightarrow{\textcircled{2}} 11 - x = 9 + 6\sqrt{x + 6} + x + 6 \xleftrightarrow{\textcircled{3}}$$

$$\xleftrightarrow{\textcircled{3}} -4 - 2x = 6\sqrt{x + 6} \xleftrightarrow{\textcircled{4}} -2 - x = 3\sqrt{x + 6} \xrightarrow{\textcircled{5}} x^2 + 4x + 4 = 9(x+6) \xleftrightarrow{\textcircled{6}}$$

$$\xleftrightarrow{\textcircled{6}} x^2 - 5x - 50 = 0 \xleftrightarrow{\textcircled{7}} (x + 5)(x - 10) = 0 \xleftrightarrow{\textcircled{8}} x = -5 \text{ or } x = 10$$

$\sqrt{11 - (-5)} = 3 + \sqrt{-5 + 6}$ is true

$\sqrt{11 - 10} = 3 + \sqrt{10 + 6}$ is false

$\therefore\ \sqrt{11 - x} = 3 + \sqrt{x + 6} \longleftrightarrow x = -5$

Steps ② and ⑤ show the non-equivalence of ① and ⑧.

GEOMETRY

A FLOW PROOF APPROACH

Robert McMurray

1976

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INTRODUCTION

or DON'T YOU DARE SKIP THIS!

You are about to begin the study of a subject which is different in nature from any other you have studied. As you would expect, it will relate closely to your knowledge of algebra and previous mathematics, but it will also relate very closely to your knowledge of English grammar. -- Don't frown like that!

The study of geometry has many purposes and objectives which will become evident as you weave your way through the course. But at this point, our main concern is with your approach to the study of geometry. Your very first habits and attitudes can make you or break you in the later stages.

First of all, it is a reading course. Accept this and try to enjoy it, as it is a truly fascinating subject. But do not attempt to read it as a novel. Many leading mathematicians have admitted to rereading certain passages in math textbooks many times in order to grasp a particular point. If this seems time consuming, experience has shown us that in the long run it is definitely a time saver.

There will be some memorization involved, but try to grasp concepts rather than just memorizing words. Hopefully, this will be aided by the use of the word or relationship as you work with it.

Geometry is a sequential course of study. That is, just as in algebra, mastery of each topic or idea is essential to the understanding of a later topic or idea. You cannot build the tenth floor of a skyscraper unless there was a solid seventh floor.

Geometry is not a spectator sport. You can no more learn geometry by watching someone else do it than you can learn how to swim by watching someone else swim. You've got to jump right in and start using your muscles. Brain muscles, that is.

Geometry can at times be frustrating. If you are able to understand every concept without help and work every problem without help, you are truly exceptional and the rest of us will probably be reading your textbook in a few years. But for the vast majority, there will undoubtedly be stumbling blocks. The personal rewards of doing a difficult problem or proof by yourself are immense. But when you have convinced yourself that you have done everything within your power, then be satisfied with your efforts and look for help.

And finally, the great Italian astronomer, Galileo, said "No man can teach another anything, he can only help him to find it within himself". Your teacher will be of a great deal of help to you, but you should think of your teacher as helping you to learn. The ultimate process is going to have to take place directly between your ears.

CHAPTER 1 - SOME HISTORY OF GEOMETRY

or WHERE DID IT ALL COME FROM?

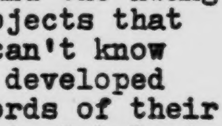
The Beginning

Would you believe that geometry may have some of its origins in attempts to collect taxes? One version refers to geometry as a "gift of the Nile". The story goes that in the fourteenth century B.C. King Sesostris had divided the land among all Egyptians so that each received a rectangle of the same size and was taxed accordingly. But the Nile overflowed its banks each year and if a man lost any of his land because of this, he was to report the loss to the Pharaoh. An overseer was then sent to measure the land and adjust the tax. From the words geo, meaning earth, and metron, meaning measure, geometry came into being.

Actually, knowledge of geometry was known to have existed in both Egypt and Babylonia long before the fourteenth century B.C. The great river basins of the known world produced some amazing results in engineering and agriculture. Not only the Nile, but the Tigris and Euphrates of Mesopotamia, the Indus and Ganges of south-central Asia, and the Hwang Ho and Yangtze of eastern Asia developed many projects that most certainly required practical geometry. We can't know for certain just how much geometry was known and developed in each of these regions because many of the records of their accomplishments have surely been lost. However, in Babylonia the use of baked clay tablets, and in Egypt the use of papyrus have given us some remarkably well preserved records of their deeds. In the other areas unfortunately more perishable materials were used and less is known about this same period in time.

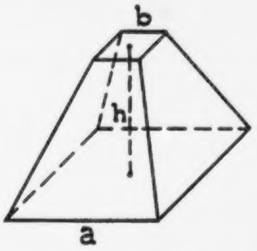
For the period from 2000 to 1600 B.C. there is evidence that the Babylonians were familiar with the general rules for computing the areas of many of the simple figures. They computed the circumference and area of a circle using $\pi = 3$. They also were apparently familiar with some geometric relationships that you are not yet familiar with but will encounter in this course.

The main sources of information about ancient Egyptian geometry are the Moscow papyrus, dating from about 1850 B.C., and the Rhind papyrus, dating from about 1650 B.C. These documents contained 110 problems in mathematics, 26 of which were geometric. They took the area of a circle to be equal to the area of a square on 8/9 of the diameter. They are not the same but just a glance at the figure shows them to be pretty close.



They used a number of other formulas for areas and volumes, but the most remarkable was a numerical example of a correct formula for the volume of a frustrum of a square pyramid as shown.

$$V = \frac{h(a^2 + ab + b^2)}{3}$$



The proof of this formula requires some advanced mathematics which the Egyptians did not have, so their application of it represents an astonishing discovery. So much so that one historian has referred to this achievement as "the greatest Egyptian pyramid".

Speaking of pyramids, the great pyramid of Gizeh was constructed about 2900 B.C. It supposedly involved a relative error of less than 1/14,000 in the lengths of the square sides and a relative error of less than 1/27,000 in the right angles at the corners. Of course, it took 100,000 laborers and 30 years to build, so it was mostly an accomplishment in patience and engineering. But undoubtedly, some knowledge of geometry was involved.

Almost all of the geometry of this ancient period was developed as a result of a practical need. As amazing as their discoveries were, the only consideration given to geometry was its usefulness. A very different and dramatic change in this viewpoint took place in the next phase of this magnificent story as we shall see.

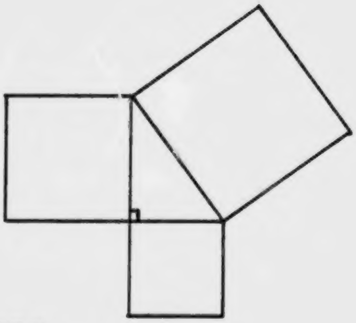
The Greeks and Demonstrative Geometry

The baton in the geometry relay was next handed to the Greeks and during the period from about 600B.C. to 300 B.C., they carried it to the point that is very close to our viewpoint on it today. At this stage we can begin to name the individuals that were responsible for the development of geometry. Throughout all of history you will not find any more fascinating stories than those associated with the people that developed mathematics at all its stages. To appreciate the role of the Greeks in this saga, we must recognize the fact that they were very much concerned about philosophy and reason. They were the first people as a group to seriously consider the question "why?" rather than just "how?".

Apparently the first of this new breed was Thales of Miletus in Asia Minor. He was considered to be one of the "seven wise men" of antiquity and many stories are told about him. He was a shrewd merchant, a statesman, engineer, philosopher, as well as mathematician. He was also the first known individual to use logical reasoning to show that a geometric relationship was true. One of those relationships that you will use later in this course is as follows: If two sides of a triangle are equal in length, then the angles opposite those two sides are equal in measure. This may not seem like a very earthshaking statement, but the fact that Thales was the first to "demonstrate" that it was true by logical reasoning was revolutionary. This was the beginning of the deductive reasoning process.

For the next 300 years the Greeks were absorbed with the notion that all such truths must be demonstrated in this manner. They not only had a complete disregard for practicality, but actually looked down upon any attempts to make practical use of their discoveries. They found geometry to be the ideal subject to relate to their philosophy of logic and reason. In this manner it became the study of "pure mathematics". That is, the study of mathematics for its own sake. You would find this to be the case in many of the astonishing developments in mathematics throughout its history. It has actually been a part of the culture of civilized man just as art, music, and drama. But an important lesson here is one you should remember well. Although a tremendous amount of mathematics has been discovered in the "pure" sense, valuable applications have almost invariably been found for these discoveries. Such applications may have come centuries later, but this phenomenon is still taking place today just as it has since Thales.

Meanwhile, back at the Mediterranean, a Greek named Pythagoras came along about 50 years after Thales. After a lot of travel and study, he founded a school at Crotona, in southern Italy. It developed into a closely knit brotherhood with secret rites and observances and was devoted to the study of philosophy, mathematics, and science. The Pythagorean school continued the emphasis on systematizing geometry, and the idea of developing a chain of theorems, each derived from earlier ones, emerged. You have surely heard of the Pythagorean theorem, by one name or another, relating the three sides of a right triangle. This relationship was apparently known to the ancient Babylonians, but the first general proof of it came from the Pythagoreans. They conceived the problem as follows.



The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two legs.

This was referring to the areas of the three squares as shown. It has become one of the most powerful theorems in mathematics and has a rich history to the present day. You will see more of it later. Because of the mystery surrounding the Pythagoreans, it is not known whether the proof came from Pythagoras himself or one of the members of the group. Well over 300 proofs have now been classified for the theorem.

The Pythagoreans exerted influence in mathematics for about 200 years. This, together with a new historical development, merged into the next phase of geometry still involving the Greeks.

Euclid and Alexandria

When Alexander the Great set out to conquer much of the civilized world, he established a string of new cities in his path. This is how Alexandria, in Egypt, came into being. When Alexander died, Alexandria eventually came under the rule of one of his military leaders, Ptolemy, about 306 B.C. The port city was ideally located at the junction of Asia, Africa, and Europe. Under Ptolemy, it was to become one of the most unique centers of intellectual activity in the history of mankind. He established the University of Alexandria including a library which contained over 600,000 papyrus rolls. He invited some of the greatest intellects of the entire Mediterranean region to the University, supporting them with grants. Most of them came from Greece and it became one of the most notable collections of scholars ever assembled.

The man who made a lasting imprint on mathematics was Euclid. Very little else is known about him, but his writings were the most influential in the entire history of mathematics. About 300 B.C., he organized all the knowledge of geometry from the previous 300 years in a monumental work called the Elements. This consisted of 13 books and covered a wide range of mathematics, but is primarily cited for its effect on the study of geometry. Euclid started with a few very simple assumptions and definitions and from these he "deduced" a logical sequence of 465 propositions. It has been stated that only the Bible has been more widely read than Euclid's Elements. In a very basic way, it is the geometry that you are about to study. It is primarily the approach itself, which we will later call the postulational approach, which has become entrenched in all mathematical thought since that time.

A story is told that Ptolemy approached Euclid and asked him for a short cut to geometric knowledge. Euclid's reply was "There is no royal road to geometry". It is a reply that you may well heed yourself as you travel this centuries old path.

Although Euclid's work is of the greatest significance for our present study, many great mathematicians were either professors or students at the University of Alexandria. The greatest of these was Archimedes, frequently heralded as one of the three greatest mathematicians of all time. Another was Apollonius who earned the name "The Great Geometer". If you enjoy stimulating mental activity, read more about these men in a history of mathematics book, along with Eratosthenes, Heron, Diophantus, and Pappus. Pappus followed Archimedes by more than 500 years and was considered the last of the original creators of mathematics among the Greeks. The University struggled with adversity until finally, in 640 A.D., it was burned by the Moslems. Thus ended over 900 years of the glorius Greek chapter in the history of mathematics.

Up Till Now

Not much went on in mathematics, or anything else of a creative nature for that matter, until about 1200 A.D. This was Europe's Dark Ages. The mathematics that did take place during this period was mostly by the Hindus and Arabians. The Hindus made some noteworthy accomplishments, but not much in geometry. We are indebted to the Arabians primarily for the preservation and transmission of a great deal of mathematics. As a point of interest, the famed Arabian poet, Omar Khayyam, produced some outstanding work in finding geometrical solutions to some higher degree equations.

The next significant developments in geometry began to emerge with the discoveries of "different kinds" of geometry. In 1639, a French engineer and architect named Desargues produced a projective geometry motivated by the needs of artists and architects for a deeper theory of perspective. The work didn't gain a great deal of attention until well after the death of Desargues.

About this same time, one of the most important developments in all mathematics appeared from two Frenchmen, René Descartes and Pierre de Fermat. This is analytic geometry, an approach that you will be exposed to later in this course as coordinate geometry.

The continual examination of Euclid's work resulted in some very surprising and unexpected outcomes. In 1733, the seeds were sown for a fresh viewpoint and about 100 years later several Non-Euclidean geometries were shown to exist. We'll comment on this more in our unit on parallels.

Without Euclid's Elements we can only speculate as to what would have happened in geometry. It has been both blessed and blasted, and some of the dissatisfaction with certain aspects of it has led to new variations, new content, and new approaches which are still ocurring to this day. We can't begin to touch upon all of it in this course, but it is hoped that this discussion will give you a frame of reference for the subject and a good running start.

EXERCISE SET 1.1

1. Why has geometry been referred to as a "gift of the Nile"?
2. What value did the early Babylonians use for π?
3. Use the Egyptians method of approximating the area of a circle with diameter 18 inches. (The area of a square with side s is s^2.) Compare this with the area of the circle using the formula $A = \pi r^2$ with $\pi = 3.14$.
4. In what way did the early Greeks viewpoint on mathematics differ from the Egyptians?
5. Why was Thales important in the history of geometry?
6. The study of mathematics for its own sake is called __________________.
7. The group that influenced Greek mathematics until about 300 B.C. were the __________________.
8. About 300 B.C. the center of learning was located at the city of __________________.
9. The prominent Greek writing that influenced mathematical thought for centuries was ________________________.
10. Name three developments in geometry that began to emerge in the 17th and 18th centuries.

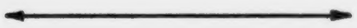
CHAPTER 2 - BASIC CONSTRUCTIONS
or DON'T GOUGE A HOLE IN THE PAPER!

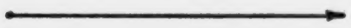
Six Basic Constructions

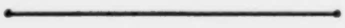
We're going to consider just a few of the basic constructions at this point and later in the course we'll look at some more involved construction problems. You may or may not be familiar with those presented here, but they should be available to you so that throughout the course you may use them when they are required, or just whenever you would like. One of the most enjoyable parts of geometry is the experimenting a person can do with the tools of construction.

We must first go back to the Greeks. They established the ground rules for construction that have essentially been followed ever since. The allowable instruments are a compass and an unmarked straightedge. A compass is sometimes called a pair of compasses (like a pair of scissors). It has been shown that with only two marks on a straightedge, constructions are possible that would not be possible without them. The original rules of the game in Euclid's time required a compass that "collapsed" when either point was removed from the paper. Fortunately, we now allow the opening to remain fixed when moving the compass. We will also allow you to use a ruler as a straightedge except that you can't use the marks on it.

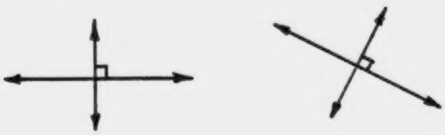
A brief review of some of the knowledge you already have about geometry will be helpful here, but we will also consider these same items more formally later in the course.

Line 

Ray 

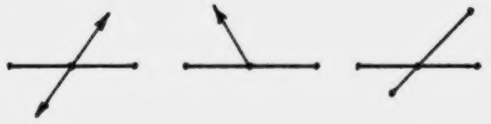
Segment 

Two lines are perpendicular if they form right angles.

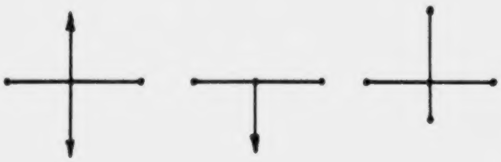


The midpoint of a segment divides it into two segments of equal length.

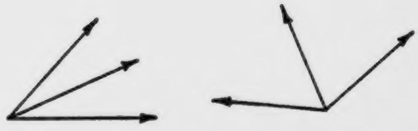
A bisector of a segment is a line, ray, or segment that intersects the given segment at its midpoint.



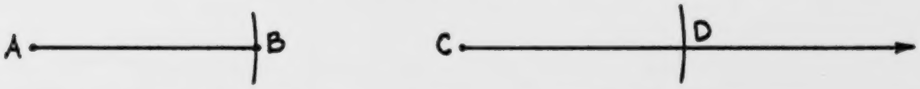
The perpendicular bisector of a segment is a line, ray, or segment that is perpendicular to the given segment at its midpoint.



The bisector of an angle is a ray that divides the angle into two angles of equal measure.



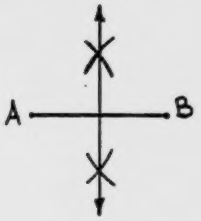
Construction 1: Construct a segment equal in length to a given segment.

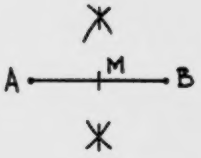


With A as center, strike an arc of a circle through B. With the same radius and C as center, strike an arc intersecting the ray at some point. Call it D. The segment from C to D is the desired segment equal in length to the given segment from A to B.

Construction 2: Construct the perpendicular bisector of a segment.

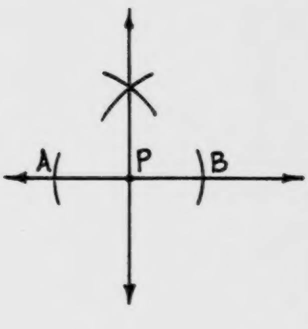
With A as center and a convenient radius greater than half the distance from A to B, strike arcs on both sides of the segment. With the same radius and B as center, strike arcs intersecting the first two. The line containing these two points of intersection is the perpendicular bisector of the given segment. If you were asked to construct the midpoint of a segment, you would do the same thing only you may modify it as shown to reduce clutter. M is the midpoint.





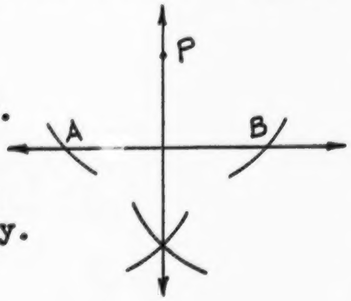
Construction 3: Construct a perpendicular to a line at a point on the line.

With the given point P as center and any convenient radius, strike two arcs intersecting the given line in two points. Call them A and B. With A as center and a convenient radius greater than the distance from A to P, strike an arc on one side of the given line. With the same radius and B as center, strike an arc intersecting the arc drawn from A. The line containing this point and P is perpendicular to the given line at P.



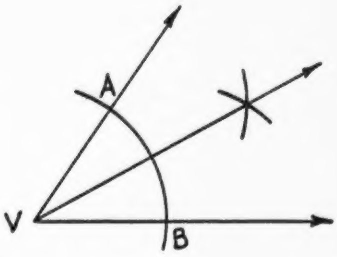
Construction 4: Construct a perpendicular to a line through a point not on the line.

With the given point P as center and a convenient radius, strike two arcs intersecting the given line in two points. Call them A and B. Now continue as in construction 3. It's best to strike the last two arcs on the opposite side of the given line from P for the sake of accuracy.

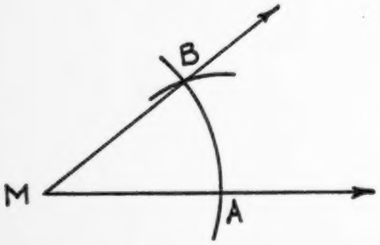


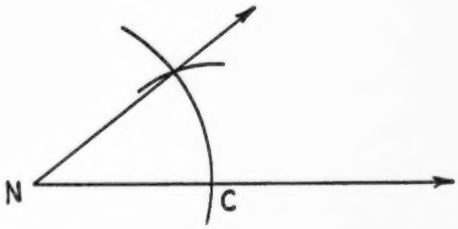
Construction 5: Construct the bisector of a given angle.

With V as center and any convenient radius, strike an arc intersecting both sides of the given angle. Call the points of intersection A and B. With A as center and a convenient radius greater than half the distance from A to B, strike an arc in the interior of the angle. With the same radius and B as center, strike an arc intersecting the arc from A. The ray containing this point and V is the angle bisector.



Construction 6: Construct an angle equal in measure to a given angle.



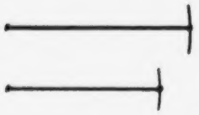


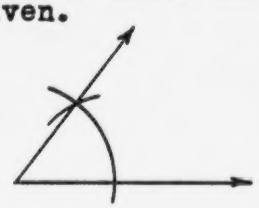
Let the angle at M be the given angle. First draw a ray which will be one side of the constructed angle. Call the endpoint N. With M as center and any convenient radius, strike an arc intersecting both sides of the given angle. Call the points of intersection A and B. With the same radius and N as center, strike an arc intersecting the ray at a point. Call it C. With A as center, strike an arc through B. With this same radius and C as center, strike an arc intersecting the arc from N. The ray containing this point and N will complete the desired angle.

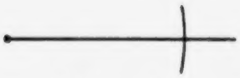
You are still a long way from knowing a lot about geometric constructions. Hopefully this will come in a later chapter. But there are many things you can do with these basic constructions, and you will have plenty of opportunities to practice them.

Example:

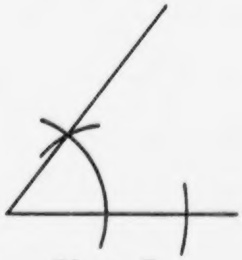
Construct a triangle in which two sides and the angle formed by these two sides are given.



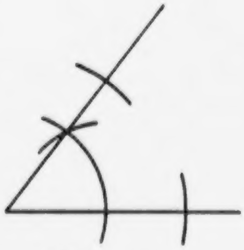




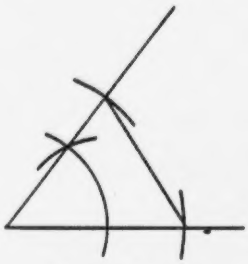
Step 1



Step 2



Step 3



Step 4

The steps shown would, of course, be in a single figure and would appear as in step 4. The last step, that of drawing a segment joining two points, may be considered a basic construction in itself. Keep these instruments handy and don't be afraid to experiment with them even when you're not required to.

EXERCISE SET 2.1

Draw figures that look like those shown here and perform the following constructions. Do not mark in your book.

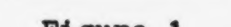


Figure 1

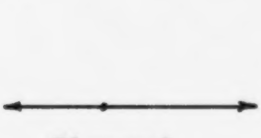


Figure2

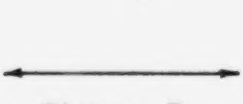


Figure 3

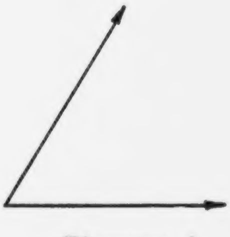
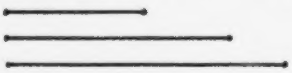
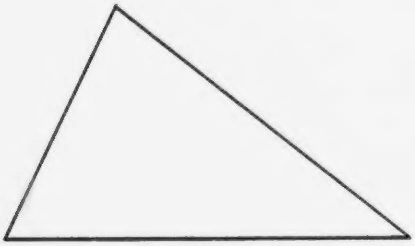


Figure 4

1. Construct a segment equal in length to Figure 1.
2. Construct the perpendicular bisector of Figure 1.
3. In Figure 2, construct a perpendicular to the line at the given point.
4. In Figure 3, construct a perpendicular to the line through the given point.
5. Bisect the angle in Figure 4.
6. Construct an angle equal in measure to Figure 4.
7. Draw three segments approximately the lengths of these. Construct a triangle whose sides are equal in length to the three given segments.



8. Draw a triangle that looks like the one shown only larger. Construct a perpendicular from each vertex to the opposite side.



CHAPTER 3 - INDUCTION

or HOW MANY TIMES DO I HAVE TO DO THIS?

A Discovery Process

Have you ever noticed how the sound of a horn on a car coming toward you has a higher pitch than the sound of the same horn when it's going away from you? The first time you notice it, you may not think much about it. The next time it happens with a different car, you may begin to wonder. When it happens with other horns and other moving sounds such as a train whistle, you probably begin to think there's a principle involved with all similar moving sounds. You are using inductive reasoning. Aren't you proud of yourself?

Any conclusion reached on the basis of the observations of a number of cases or the outcomes of a number of experiments is the result of inductive reasoning.

This is undoubtedly the type of reasoning used by the early mathematicians of Egypt and Babylonia. They discovered many relationships that they couldn't prove just because of their tenacity in staying with a problem. They were aware of the relationships of the sides of a right triangle long before Pythagoras. But sometimes their conclusions were right and sometimes they were wrong. Induction is a very valuable tool if we recognize it for what it is. Regardless of the number of cases observed, any conclusion reached in this manner can be considered, at best, a highly probable, or plausible, conclusion. A man may conclude that all birds can fly because he has never seen an ostrich. A person declares "It always rains on the fourth of July" because it has rained for two years in a row on the fourth of July.

Induction is an essential process in both science and mathematics. You've probably used it without realizing it. A grade school student observes that in ten problems of adding two odd numbers the sum is even, and concludes that when two odd numbers are added the sum is always even. His conclusion is true and can be proved. The process of proving it is called deduction and that is what our course is mostly about. In general we can say that induction is a process of discovery, and deduction, which we will look at later, is a process of proof.

Let's look at some examples together before turning you loose on your own.

In 1742, a mathematician named Goldbach considered the problem of writing even numbers as a sum of prime numbers as follows.

$6 = 3 + 3$
$8 = 3 + 5$
$10 = 3 + 7$
$12 = 5 + 7$
$14 = 3 + 11$
$16 = 5 + 11$
$18 = 5 + 13$
$20 = 3 + 17$
$22 = 11 + 11$
$24 = 11 + 13$
$26 = 7 + 19$
$28 = 11 + 17$

He came to the conclusion inductively that every even number greater than 4 can be represented as the sum of two odd prime numbers. Do you think it's true? Try some more if you like. Just so you don't think everything in mathematics is cut and dried, there has never been an even number found for which this is not true. At the same time, it has never been proved that it must be true for all even numbers. It is known as Goldbach's conjecture, and is one of many still unanswered problems in mathematics.

Some examples of induction involve the recognition of a pattern in the cases that are considered. Look at this one.

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1+3 &= 4 = 2^2 \\ 1+3+5 &= 9 = 3^2 \\ 1+3+5+7 &= 16 = 4^2 \\ 1+3+5+7+9 &= 25 = 5^2 \\ 1+3+5+7+9+11 &= 36 = 6^2 \\ 1+3+5+7+9+11+13 &= 49 = 7^2 \end{aligned}$$

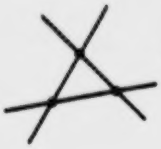
Do you recognize the pattern of finding the sum of consecutive odd integers? If the pattern continues, use your smart cells and give the sum of the first ten consecutive odd positive integers without adding them up.

$$1+3+5+7+9+11+13+15+17+19 = ?$$

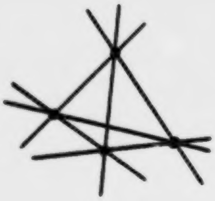
Now add them up and see if you get the same thing. If you don't, then put your teacher to work. It can be proved that this pattern does continue for all cases.

Consider this problem from geometry.
Given a set of points, no three of which lie on one line, how many lines can be drawn through each pair of points?

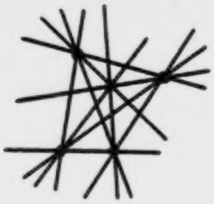
2 points
1 line



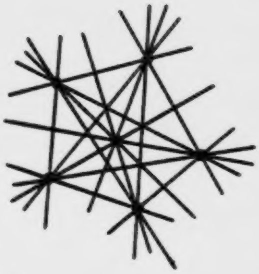
3 points
3 lines



4 points
6 lines



5 points
10 lines



6 points
15 lines (whew!)

Sometimes recognizing a pattern isn't easy. Maybe it would help if we could relate each case to the previous case.

2 points,	1 line	
3 points,	3 lines	$(3 = 1 + 2)$
4 points,	6 lines	$(6 = 3 + 3)$
5 points,	10 lines	$(10 = 6 + 4)$
6 points,	15 lines	$(15 = 10 + 5)$

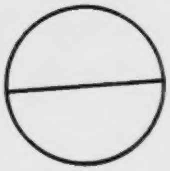
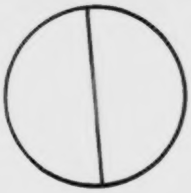
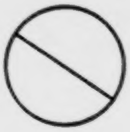
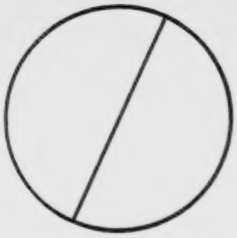
Do you notice how the number of lines in each case is related to the numbers in the previous case? If this pattern continues, we should be able to predict --

7 points, should be $15 + 6 = 21$ lines
8 points, should be $21 + 7 = 28$ lines

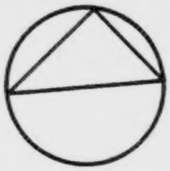
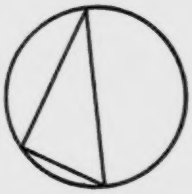
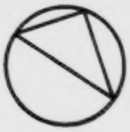
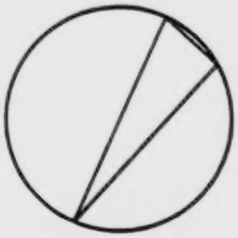
Maybe you're smart enough to see why this pattern would continue. In any case, it can also be proved.

In the last two examples, the complete problem would require a general statement of the conclusion that was discovered by induction. But you will run into this in a later course if you continue your studies in mathematics.

As an example of induction in our present course, look at the four circles below with a diameter drawn in each one.

Now we'll draw segments from the endpoints of that diameter to a random point on the circle.

Do you notice anything special about the angle formed opposite the diameter? If your eyesight is normal, it should appear to be a right angle (a square corner), a nice discovery.

When experimenting in this manner, be sure to draw each figure different, and yet retaining the basic characteristics of the problem. This is the only way to make a good generalization that would be true for all such figures. An exploratory spirit is most helpful in geometry.

EXERCISE SET 3.1

1. Drop five thumbtacks on a hard flat surface and record the number of points that are up and the number of points that are down. Repeat this ten times. State a conclusion indicating the likelihood of the point being up or down.

2. 12, 15, 24, 45, 72, 87, 141, 504, 2874, and 17202 are all divisible by 3 (verify this). Find the sum of the digits of each number. (Example: 12, $1+2=3$) Without dividing first, do you think 717 is divisible by 3? Why or why not? Make a general statement about numbers divisible by 3.

3. Observe the following pattern.

$$\begin{aligned} 2 &= 2 &= 1 \cdot 2 \\ 2+4 &= 6 &= 2 \cdot 3 \\ 2+4+6 &= 12 &= 3 \cdot 4 \\ 2+4+6+8 &= 20 &= 4 \cdot 5 \\ 2+4+6+8+10 &= 30 &= 5 \cdot 6 \end{aligned}$$

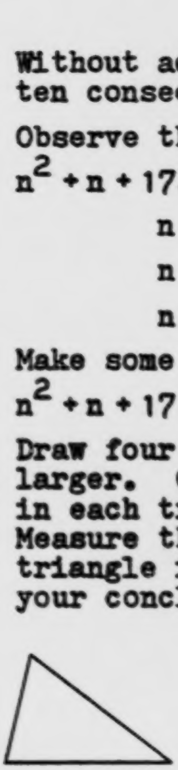
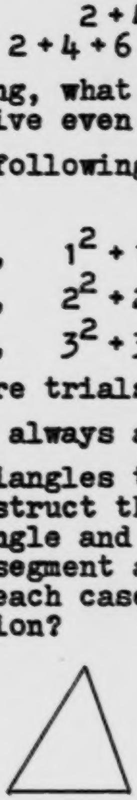
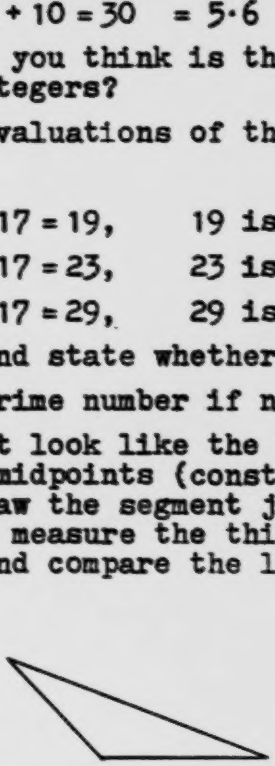
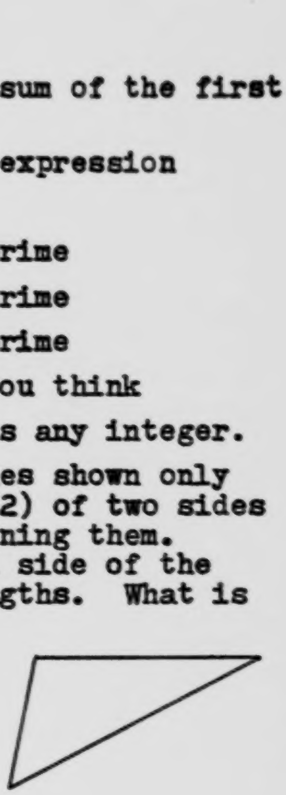
   Without adding, what do you think is the sum of the first ten consecutive even integers?

4. Observe the following evaluations of the expression n^2+n+17.

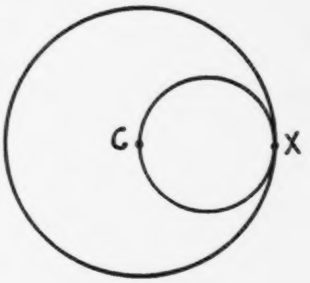
   $n=1$, $1^2+1+17=19$, 19 is prime
   $n=2$, $2^2+2+17=23$, 23 is prime
   $n=3$, $3^2+3+17=29$, 29 is prime

   Make some more trials and state whether you think n^2+n+17 is always a prime number if n is any integer.

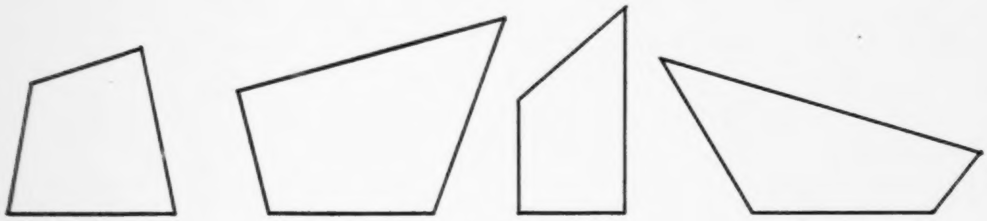
5. Draw four triangles that look like the ones shown only larger. Construct the midpoints (const. 2) of two sides in each triangle and draw the segment joining them. Measure the segment and measure the third side of the triangle in each case and compare the lengths. What is your conclusion?

6. In the figure shown, the smaller circle contains the center, C, of the larger circle and one point, X, of the larger circle. Draw a similar figure and from point X draw several segments to other points on the larger circle. For each of these segments, measure the lengths of the parts inside the smaller circle and outside the smaller circle and compare these lengths. What is your conclusion?



7. Draw four figures that look like the ones shown only larger. Construct the midpoint of each side in each figure and join them consecutively within each figure. Observe the resulting figures and state a conclusion.



8. In exercise 4, if you concluded that $n^2 + n + 17$ is always prime when n is any integer, show that your conclusion is false by letting n = 17. (Hint: recall factoring from algebra)

CHAPTER 4 - SETS AND VENN DIAGRAMS

or WHERE DO YOU BELONG?

Set Notation and Operations

You probably recall the idea of a set from your previous math classes. We will review some of those ideas now and discuss how they may apply to geometry. Just remember that a set is merely a collection of objects, which we generally call the elements of the set.

The sets that you worked with in algebra normally contained elements that were numbers. For example, if we want to consider the set of prime numbers less than 15, we may name it with a capital letter, such as P, or we may designate it by writing the elements within braces as follows.

$\{2, 3, 5, 7, 11, 13\}$

We may then write $P = \{2, 3, 5, 7, 11, 13\}$.
If we want to state that a particular number belongs to the set, we write for example $5 \in P$, which is read "5 is an element of P". Say that we name two more sets as follows.

$E = \{2, 4, 6, 8\}$
$Q = \{3, 5, 7\}$

Notice that every element in Q is also in P. We say that set Q is a subset of set P and we write $Q \subset P$.

We accept the idea of a set which contains no elements, and we call it the empty set, designated as $\emptyset$. For example, the set of all even prime numbers greater than 3 is the empty set.

The two basic operations with sets are intersection and union.

The intersection of two sets is the set of all elements which belong to both the first set and the second set. The symbol for intersection is $\cap$. Using the sets P, E, and Q named above,

$$P \cap E = \{2\}$$
$$P \cap Q = \{3, 5, 7\}$$
$$E \cap Q = \emptyset$$

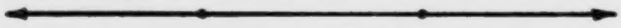
The union of two sets is the set of all elements which belong to either the first set or the second set or both. The symbol for union is $\cup$. Using the same sets,

$$P \cup E = \{2, 3, 4, 5, 6, 7, 8, 11, 13\}$$
$$P \cup Q = \{2, 3, 5, 7, 11, 13\}$$
$$E \cup Q = \{2, 3, 4, 5, 6, 7, 8\}$$

Notice that in the subset relation, $Q \subset P$, we have
$P \cap Q = Q$ and $P \cup Q = P$
A little thought should verify the fact that this is a general rule whenever one set is a subset of another.

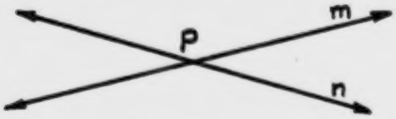
In geometry the basic elements of the sets we work with are points, lines, and plenes. Other sets may consist of figures composed of these three kinds of elements.

In the figure below, the segment determined by the two points shown is a subset of the entire line.



The operation of intersection in geometry sets is consistent with your concept of intersection in the usual sense. In the next figure, if we call the lines m and n, and their point of intersection P, then we can write

$m \cap n = P$



By the definition of union,

$m \cup n$ is the figure itself.

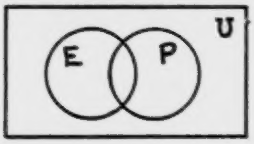
In geometry, if the intersection of two sets is the empty set, we normally say they do not intersect.

As another example, a triangle may be considered as the union of three segments. There are times when the language of sets will be quite useful in making definitions and stating relationships. Make sure you're familiar with it.

Venn Diagrams

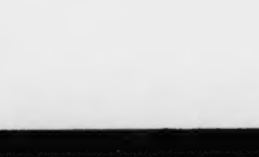
A useful way of looking at set relationships and operations is with a Venn diagram, appropriately named for its inventor, a nineteenth century English mathematician named John Venn. We draw a rectangle and consider it as containing all possible elements we may encounter in a particular investigation. We call it the universal set. Within it we may show any number of subsets of the universal set and their relationship to one another.

Let U = {all numbers}
E = {all even numbers}
P = {all prime numbers}

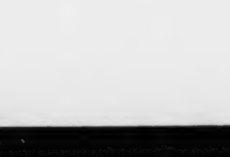


The size of the circles is of no significance nor is the size of the overlapping region if there is an intersection. In this case, the intersection would contain the single number 2. We may indicate the operations by shading that region of the set.

$E \cap P$

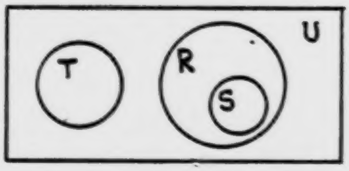


$E \cup P$



Now an example from geometry.

Let $U = \{\text{all geometric figures}\}$
$T = \{\text{all triangles}\}$
$R = \{\text{all rectangles}\}$
$S = \{\text{all squares}\}$



If you didn't already know it, every square is a rectangle. The diagram shows the relationship $S \subset R$. If two sets, such as T and R, do not intersect, they are called disjoint sets. We can write $T \cap R = \emptyset$.

Now, if you begin to get filled up with sets, don't forget to say Venn.

EXERCISE SET 4.1

Given the following sets.

$A = \{r, i, g\}$
$B = \{h, i, t\}$
$C = \{w, o, r, n\}$
$D = \{o, f, t, e, n\}$
$E = \{\text{all even numbers}\}$
$F = \{\text{all multiples of } 3\}$
$G = \{\text{all prime numbers}\}$

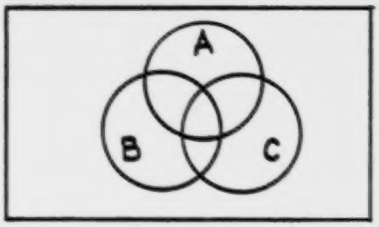
1. $A \cup B =$
2. $C \cap D =$
3. $B \cap C =$
4. $E \cap F =$
5. What is the smallest positive integer that is not an element of $(E \cup F) \cup G$?

Draw a Venn diagram to illustrate the relationship between each of the following sets.

6. $U = \{\text{all animals}\}$
   $F = \{\text{all animals that fly}\}$
   $B = \{\text{all birds}\}$
7. $U = \{\text{all numbers}\}$
   $P = \{\text{all prime numbers}\}$
   $S = \{\text{all squares of integers}\}$
8. $U = \{\text{all people}\}$
   $S = \{\text{all singers}\}$
   $M = \{\text{all musicians}\}$
   $W = \{\text{all songwriters}\}$
9. $U = \{\text{all people}\}$
   $A = \{\text{all athletes}\}$
   $B = \{\text{all baseball players}\}$
   $L = \{\text{all left handed people}\}$

Draw the figure shown for each example and shade the region indicated.

10. $A \cup B$
11. $B \cap C$
12. $(A \cap B) \cap C$
13. $(C \cup B) \cup A$
14. $(A \cup B) \cap C$
15. $(A \cap B) \cup C$



CHAPTER 5 - LOGIC AND DEDUCTIVE REASONING

or IF SO, THEN WHAT

Reasoning and Deduction

Now you think you're a logical thinker, don't you? Well, how would you define the word logic anyway? Sort of like common sense, huh? Whatever that is. Say you are given the following information. "Uncle Casper lives in a haunted house with 5 rooms and 33 windows." Now consider whether the following statements seem to be true or false.

1. Uncle Casper is a nut.
2. Every room in the house has at least one window.
3. At least one room has an odd number of windows.
4. All the windows are not in one room.

Whatever answer you gave on 1, 2, and 4 is acceptable because the directions said consider whether they "seem" to be true or false and that is up to you. But you may have considered each of these three to be reasonable statements that are probably true. On number 1, your reasoning might be called intuition, or you just felt that it must be true. 2 and 4 just seem like common sense, but you'll have to agree that they could be false. Right? Now on 3, you'd better say true for sure or we have a problem. All your answers are based on some kind of reasoning process, but 3 is different because we can prove conclusively that at least one room must have an odd number of windows. Maybe you can convince yourself of that at this point, but later in the course you will have this as a problem to prove formally.

The type of reasoning we are most interested in here is called deductive reasoning. It is the type in which we can prove our conclusions are true "because of" the information we have available. The nature of geometry makes it an ideal subject for the study of the deductive reasoning process, while at the same time learning some of the fascinating aspects of geometry itself. But before we look at deductive reasoning in geometry, we will study the process in some non-mathematical situations. This is both a means to an end and an end in itself, as it may prove to be one of the most valuable units in the course. You may even find yourself amazing your friends and confounding your parents with your superlogical approach to mind-bending situations.

Consider the following example.

Clifton Buxton III says, "Oh, if I just knew how to dance, I'd sweep sweet little Samantha Jane right off her feet." Now Cliff, being a fairly bright young man, decides to take dancing lessons and becomes an accomplished dancer. What do you think happened?

If you're normal, you may begin to have some doubts. What if Samantha Jane doesn't know how to dance? Or what if she's going steady with another guy? Or what if she just can't stand the big slob? Actually, these questions are of no concern to us right now. We are interested in conclusions that are valid or invalid rather than conclusions that are true or false.

Let's slightly restate Cliff's statement.

"If I knew how to dance, then I'd sweep S.J. off her feet."

This whole thing is one single statement. We call it a conditional statement. Within it there are two separate statements.

"I knew how to dance." This is called the hypothesis of the conditional statement.

"I'd sweep S.J. off her feet." This is called the conclusion of the conditional statement.

Notice that each of the three statements we've written is a complete grammatical sentence in itself. This is important. Now pay attention. If the hypothesis of an accepted conditional statement is true or becomes true, then the conclusion of the conditional statement is a valid conclusion because it "follows from" the first two statements. Since Cliff accepted his first statement as true and since he did learn how to dance, then he can logically conclude "I'll sweep S.J. off her feet." His conclusion is valid whether it's true or not. This is deductive reasoning. The trick is to make sure that our conditional statements are completely acceptable in order to arrive at conclusions that are both valid and true.

EXERCISE SET 5.1

For each of the following conditional statements, write the hypothesis and the conclusion.

1. If a tree is a locust tree, then it has thorns.
2. If it snows, then I will go skiing.
3. If the lake is calm, then the fish will bite.
4. If Sam makes a B in geometry, then he will make the honor roll.

Tell whether each conclusion is valid or invalid according to the first two statements in each example.

5. If a city is in New Mexico, then it is west of the Mississippi River.
   Albuquerque is in New Mexico.
   Therefore, Albuquerque is west of the Mississippi River.

6. If a figure is a rectangle, then it has four sides.
Figure A has four sides.
Therefore, figure A is a rectangle.

7. If the bridge is out, then we will not make it to Littleton tonight.
The bridge is not out.
Therefore, we will make it to Littleton tonight.

8. If a dog is a seeing eye dog, then it is well trained.
Gretchen is a seeing eye dog.
Therefore, Gretchen is well trained.

9. If the Crawdads play, then the party will be a success.
The party was a success.
Therefore, the Crawdads played.

10. If an animal is a fish, then it can swim.
Fred Finn is a fish.
Therefore, Fred Finn can swim.

Symbolic Logic and the Implication

Since this whole business of looking at all different kinds of statements can get rather sticky, we're going to try to simplify it by looking at it symbolically.

We find it convenient in studying logic to use letters, or variables, to represent statements just as we use variables to represent numbers in algebra. Say that we want to discuss the conditional statement

"If the cat is away, then the mice will play."
Let p represent "The cat is away."
Let q represent "The mice will play."

Our conditional statement can be represented as "If p, then q." Notice that the words if and then are not part of p and q. Any conditional statement may be represented in this form and the statement p is the *hypothesis* and the statement q is the *conclusion*. A conditional statement is also called an *implication*. This may be represented symbolically as $p \rightarrow q$, and is usually read as "p implies q". It may also be read as before, "If p, then q".

Now say that we accept the entire conditional statement "If the cat is away, then the mice will play." as being true. Don't worry about mousetraps and don't worry about elderly or unhappy mice. Just accept it. Now if the cat does decide to visit Decatur for the weekend, then we can logically conclude "The mice will play".

This situation illustrates on of the most important and powerful principles of our course.

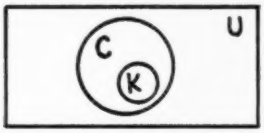
The Law of Detachment

If a conditional statement, $p \rightarrow q$, is true
and the hypothesis, p, is also true
then the conclusion, q, will be true.

That is, the statement p, by itself, is detached from the statement $p \rightarrow q$ and considered separately just as the statement "The cat is away" was detached from the original statement. We will look more in detail at the Law of Detachment later.

Conditional statements are not always stated in the "If p, then q" form. For example, "All koala bears are cute" could be stated as "If an animal is a koala bear, then it is cute", but the first form is a more comfortable form. Statements of this form are very common in geometry and lend themselves nicely to investigation by a Venn diagram. The statement says that the set of koala bears is a subset of the set of all cute animals.

If U = {all animals}
C = {all cute animals}
K = {all koala bears}



then our Venn diagram would appear as shown.
This may be helpful to illustrate the deductive reasoning process and the Law of Detachment. Let's say that we now have the information "Teddy is a koala bear". This is actually the hypothesis of our conditional statement detached for a specific case. Since that puts Teddy in set K, we can clearly see that Teddy must also be in set C and we conclude "Teddy is cute". The diagram is also helpful to guard against faulty reasoning that sometimes plagues students of geometry and logic. Consider the statement "Freddy is a cute animal". This puts Freddy in set C but we can see that Freddy may or may not be in set K, and we can make no conclusion as to whether Freddy is or is not a koala bear. Actually, we have detached the conclusion of our conditional statement for a specific case, but this does not allow us to deduce anything about the hypothesis of our conditional statement.

Truth Tables

A complete investigation of a conditional statement may be made with a truth table in which we consider every possible combination of ways in which p and q can be true or false. This is generally done in a column form as shown. Let p and q be statements as follows.

p: My ship comes in.
q: I will be rich.

p	q
T	T
T	F
F	T
F	F

The implication formed with these statements would be

$p \rightarrow q$: If my ship comes in, then I will be rich.

Now consider the following four special cases corresponding to the four rows of the table and determine whether the above implication should be labeled true or false for each case.

1. My ship came in loaded with oil during a fuel shortage. p true, q true. Surely you'll agree that our conditional, $p \rightarrow q$, is certainly true.
2. My ship came in and rammed the pier and I am being sued for a million dollars. p true, q false. No doubt the conditional statement, $p \rightarrow q$, was very false.
3. My ship did not come in but Uncle Ed died and left me a fortune. p false, q true. We cannot label the implication, $p \rightarrow q$, false because it makes no prediction as to what will happen if my ship does not come in, so we will accept it as true.
4. My ship did not come in and I am as poor as ever. p false, q false. Just as in case 3, we cannot label the implication, $p \rightarrow q$, false, so we accept it as being true on the basis of insufficient evidence.

Cases 3 and 4 are actually agreed upon as being true simply because a false hypothesis does not allow us to conclude anything about the conclusion one way or the other. The table that we have constructed is now accepted as standard for all such statements p and q that can definitely be labeled as either true or false. Notice that the only case in which the implication is false is when the hypothesis is true and the conclusion is false.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In the previous example, p and q could be either true or false. But we can illustrate each row of the table with statements which we know to be definitely one or the other.

Row 1: If $2+3=5$, then $7+8=15$ true
Row 2: If $2+3=5$, then $7+8=10$ false
Row 3: If $2+3=6$, then $7+8=15$ true
Row 4: If $2+3=6$, then $7+8=10$ true

The Law of Detachment would be illustrated by the first row in which we have information about the two columns shown and we know from our standard table that q must also be true.

p	q	$p \rightarrow q$
T		T

EXERCISE SET 5.2

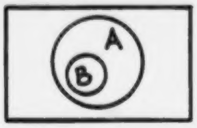
Let p and q be the statements indicated. Write the statement $p \rightarrow q$.

1. p: Today is Monday.
   q: Tomorrow is Tuesday.

2. p: Dan will play.
   q: The Rockets will win.

Draw a Venn diagram indicating the relationship between the given sets. Then write the statements illustrating the Law of Detachment.

Example: A = {all expensive cars}
B = {all Cadillacs}



If a car is a Cadillac, then it is expensive. $p \rightarrow q$ is true.
Henry owns a Cadillac. p is true.
Therefore, Henry owns an expensive car. Therefore q is true.

3. A = {all talented people}
   B = {all professional singers}

4. A = {all cities in France}
   B = {all cities in Europe}

The Law of Detachment may ocurr in situations not involving the subset relation.

Example: If it rains, then the game will be cancelled. $p \rightarrow q$ is true
It rained. p is true
Therefore, the game was cancelled. Therefore q is true

Accept the given statement as true and write the remaining statements that illustrate the Law of Detachment.

5. If Jenny swims the English Channel, then she will be famous.

6. If the birds are migrating south, then winter is on the way.

Use the truth table for the implication and state whether each of the following is true or false.

7. If New Orleans is north of St. Louis, then water flows uphill.
8. If man has gone to the moon, then man can breathe without oxygen.
9. If a triangle has three sides, then it has three angles.
10. If $4 + 5 = 8$, then $4 \times 5 = 20$.
11. If the sun rises in the west tomorrow, then I will get a job.

Other Forms of Conditionals

We noted earlier that the statement "All koala bears are cute" is a conditional statement which could be restated as "If an animal is a koala bear, then it is cute". Many statements that we'll look at later will not be stated in our convenient "If p, then q" form. But we will still need to identify the hypothesis and conclusion, so it will be helpful if we can restate it in this form even if it seems more awkward. Consider the statement.

The windows in this room are dirty.

Which of the next two statements appears to have the same meaning?

If a window is in this room, then it is dirty.
If a window is dirty, then it is in this room.

It's a good thing you said the first one. Now you should be able to identify the hypothesis and conclusion from this form. How about this one?

The sum of two odd numbers is even.

Now some students that are not as smart as you are would try to identify the hypothesis and conclusion by simply separating the statement directly as follows.

Incorrect hypothesis: The sum of two odd numbers
Incorrect conclusion: Is even

Of course you wouldn't do that because you can see that neither of these is a statement. Remember that p and q are both complete grammatical English sentences. So naturally you would rewrite the statement

If two numbers are odd, then their sum is even.

Correct hypothesis: Two numbers are odd.
Correct conclusion: Their sum is even.

Oh, your English teacher would be so proud of you!

A statement from geometry that you will prove later in this course is "The base angles of an isosceles triangle are congruent". The proof might be easier if you first restate it "If a triangle is isosceles, then its base angles are congruent". You will need to recognize the hypothesis and conclusion as the first step of the proof.

Now let's consider two statements, p and q, and form some conditional statements in different forms using them.

p: A man lives in Topeka.
q: He lives in Kansas.

The standard "If p, then q" statement would be

If a man lives in Topeka, then he lives in Kansas.

If we form the statement "If q, then p", it becomes

If a man lives in Kansas, then he lives in Topeka.

We're switching the word man from p to q here for grammatical purposes. You should agree that the first statement is true and the second is not necessarily true. We agree that if a statement is not necessarily true, it will be considered false. So obviously the two statements do not say the same thing. The statement "If q, then p" is called the converse of "If p, then q", and it may be true or false independently of the statement "If p, then q".

Study the following statements closely.

q if p: A man lives in Kansas if he lives in Topeka.
p only if q: A man lives in Topeka only if he lives in Kansas.

A close study should convince you that they are both true and therefore say the same thing as "If p, then q". Notice in particular that "p if q" and "p only if q" do not say the same thing. One is the converse of the other.

Now look at this one.

If two lines are perpendicular, then they form right angles.
If two lines form right angles, then they are perpendicular.

These are converse statements that are both true. The second one could be restated as

Two lines are perpendicular if they form right angles.

The first could be restated as

Two lines are perpendicular only if they form right angles.

We could put the two together in the single statement

Two lines are perpendicular if, and only if, they form right angles.

Whenever a statement and its converse are both true, we frequently use this single statement to combine them. This is an important characteristic of definitions that you will find it necessary to use later.

Now we can restate and summarize all these statement relationships symbolically.

"If p, then q" has the same meaning as each of the following.

q if p
p only if q
p implies q
$p \rightarrow q$

In every one of these five forms, p is the hypothesis and q is the conclusion.

If a statement and its converse are both true, we can write

If p, then q and if q, then p.
p if, and only if, q
$p \rightarrow q$ and $q \rightarrow p$

It would seem natural to write this last form as $p \leftrightarrow q$. This statement is called a biconditional and you should remember that it is always the compound statement of a conditional statement and its converse.

EXERCISE SET 5.3

When conditional statements contain such words as "all" and "every", first classify the subject in a more general category before writing in the form "If p, then q".

Example: All ducks can swim.
If a bird is a duck, then it can swim.

Rewrite each of the following in the form "If p, then q". Underline the hypothesis once and the conclusion twice.

1. Every copperhead is poisonous.
2. All clowns are funny.
3. We will go to the beach if it is sunny.
4. You may go to the movies only if you wash the dishes.
5. The diagonals of a rectangle are equal in length.

Write the converse of each of the following conditional statements.

6. If the lake is frozen, then we can go ice skating.
7. If $x+3=5$, then $x=2$.
8. If $x=3$, then $x^2=9$.
9. Two lines are perpendicular if they form right angles.
10. Every trapeze artist is well coordinated.
11. A horse will win the Kentucky Derby only if it is well trained.

12. For each case in exercises 6-11 in which the conditional and its converse are both true, write a single biconditional statement using the phrase "if and only if".

13. Complete the following truth table.

p	q	$q \rightarrow p$
T	T	
T	F	
F	T	
F	F	

Negation

Let p: The plane will arrive on time.

The symbol for the negation of p is $\sim p$, and is formally stated by prefixing "It is false that".

$\sim p$: It is false that the plane will arrive on time.

Of course, we would say the same thing much more simply.

$\sim p$: The plane will not arrive on time.

We generally associate the negation of a statement with the word not. The word "not" does ocurr at times in statements in geometry. The truth table is very simple because only one basic statement is involved. From the example given above it should be obvious that if a statement is true, its negation must be false, and if a statement is false, its negation is necessarily true.

p	$\sim p$
T	F
F	T

Sometimes the formation of a negation is not as simple as it might seem and the formal prefixed phrase "It is false that" is helpful. We can also use Venn diagrams to help in certain situations. Consider the following statements.

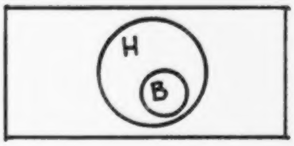
p: All bears eat honey.
q: Some bears eat honey.
r: Some bears do not eat honey.
s: No bears eat honey.

Consider the two sets as follows.

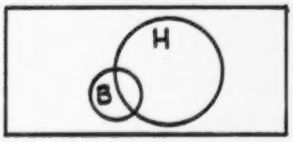
B = {all bears}
H = {all honey eaters}

Each of the four statements can be illustrated by a Venn diagram as follows.

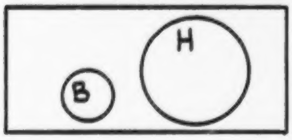
p: All bears eat honey.



q: Some bears eat honey
r: Some bears do not eat honey.



s: No bears eat honey



We will make the negations in the formal manner first and see what Venn diagram would illustrate each of them.

$\sim p$: It is false that all bears eat honey.
This is illustrated best by the second Venn diagram and the statement r.
$\sim p$: Some bears do not eat honey.

$\sim q$: It is false that some bears eat honey.
This could only mean that there aren't any bears that eat honey, or s.
$\sim q$: No bears eat honey.

$\sim r$: It is false that some bears do not eat honey.
This requires a little more thought, but doesn't it require that all bears do eat honey, or p.
$\sim r$: All bears eat honey.

$\sim s$: It is false that no bears eat honey.
Careful here. This merely requires that at least one bear eats honey, or q. Notice that it does not require p to be true.
$\sim s$: Some bears eat honey.

The word "some" is accepted as meaning "at least one".

Summarizing the four cases, we have,

$\sim p \leftrightarrow r$
$\sim q \leftrightarrow s$
$\sim r \leftrightarrow p$
$\sim s \leftrightarrow q$

Inverse and Contrapositive

Let statements p and q be as follows.

p: A flower is a rose.
q: It is fragrant.
$p \rightarrow q$: If a flower is a rose, then it is fragrant.

The converse would be

$q \rightarrow p$: If a flower is fragrant, then it is a rose.

We now form two other statements involving p and q using negations.

The inverse of $p \rightarrow q$ is
$\sim p \rightarrow \sim q$: If a flower is not a rose, then it is not fragrant.

The contrapositive of $p \rightarrow q$ is
$\sim q \rightarrow \sim p$: If a flower is not fragrant, then it is not a rose.

If you were to classify each of these four statements as true or false (no dead flowers allowed), you would probably call $p \rightarrow q$ and $\sim q \rightarrow \sim p$ both true,
and $q \rightarrow p$ and $\sim p \rightarrow \sim q$ both false.

The conditional statement and its contrapositive do actually have the same meaning. This is a good place to show how truth tables can be used to determine if two statements have the same meaning.

Recall that the truth table for the implication is as shown.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement $\sim q \rightarrow \sim p$ is an implication with $\sim q$ as its hypothesis and $\sim p$ as its conclusion. We develop its truth table in stages.

p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T			
T	F			
F	T			
F	F			

The first stage is always to assign all possible combinations of true and false for p and q. If negations are involved, make separate columns for them. The statement being investigated will always head the last column.

The second stage would be to fill in the columns for negation according to our truth table for negation.

p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Now remember from our standard table for implication, the only time the implication is false is when the hypothesis is true and the conclusion false. This ocurrs only in the second row, so we can complete the final column.

p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

When we compare the two tables side by side we notice

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim q \rightarrow \sim p$
T	T	T
T	F	F
F	T	T
F	F	T

the last columns are identical. In other words, $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are both true or both false for every combination of ways that p and q are true or false. We say that such statements are equivalent and we use the biconditional symbol,

$$(p \rightarrow q) \longleftrightarrow (\sim q \rightarrow \sim p)$$

In words: A conditional statement and its contrapositive are equivalent.

We'll find this relationship quite helpful in examining certain statements later in the course.

You should notice that for a given conditional, $p \rightarrow q$, the converse, $q \rightarrow p$, and the inverse, $\sim p \rightarrow \sim q$, bear the same relationship to one another and are also equivalent.

Symbolic logic also makes use of symbols for the important words "and" ($\wedge$) and "or" ($\vee$).

$p \wedge q$ is called the conjunction of two statements.

$p \vee q$ is called the disjunction of two statements.

Since the need for the logic involving these two statements does not arise in this course, we will not dwell on them here. If you enjoy the challenge of working with the ideas of logic in statements, there are references which will give you the full treatment.

EXERCISE SET 5.4

Write the negations of each of the following statements.

1. All birds can fly.
2. No teachers give hard tests.
3. Some children do not like candy.
4. Some students ride bicycles to school.

For each conditional statement, write the inverse and the contrapositive.

5. If the weather is clear, then the launch will be held.
6. If two lines are parallel, then they do not intersect.
7. If x is greater than 0, then x^2 is positive.
8. If two angles of a triangle have equal measure, then it is isosceles.

9. Show by truth tables that the inverse and the converse of a conditional statement are equivalent.

CHAPTER 6 - THE REAL NUMBERS AND DISTANCE
or YOU'RE MAKING SOME GOOD POINTS THERE

Undefined Terms

We are at the point where we can begin to look formally at the subject of geometry. That is, we want to develop it as a system, neatly tied together, and all wrapped up in one big beautiful package.

When Euclid was at this stage, he felt the logical approach was to make some very basic definitions, some very basic assumptions, and start to build. He was so locked in on this viewpoint that he felt even the most primitive geometric terms must be defined. We stated earlier that the basic elements in geometry are point, line, and plane. Euclid made the following definitions.

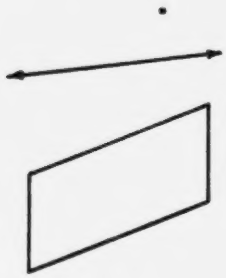
A point is that which has no part.
A line is breadthless length.
A straight line is a line which lies evenly with the points on itself.

If you didn't already know what these things were, it's not likely that these definitions would help you much. Many criticisms of Euclid's work have been made, particularly in the last 150 years, and this is one of them. But how would you define a point or a line? Sometimes the simplest ideas seem to be the hardest to define. You would certainly have to use other words whose meanings were known, wouldn't you? That means that they would have to be defined also, or we would just have to accept their meaning without definition. Well, why can't we just start with the word point and say that we know what it is without the need for a word definition? This has been found to be the most reasonable approach with all three words, point, line, and plane. We call them undefined terms.

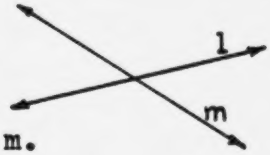
Some discussion is still necessary about the three words, however, to make sure that we do know, in fact, what they mean. Even though geometry does have wide application in physical situations, it is far more logical to study the subject in a completely abstract sense. If you think of a point as a dot, then you would have to decide how big it is. But can't you "conceive" of a point 12 inches directly in front of your nose without seeing it? (Don't cross your eyes!) Can you visualize a line from the upper corner of the room you're sitting in to the diagonally opposite lower corner? If you are a football fan, you know that the rules say a touchdown is scored when the ball crosses the plane of the goal line.

These are abstractions that you can "see" in a conceptual sense but not in a physical sense. Actually the study of geometry is greatly simplified by considering these elements in this manner.

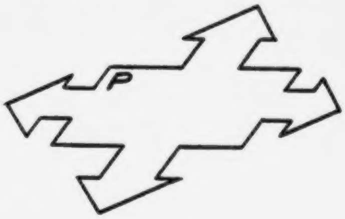
Throughout the course we'll represent point, line, and plane by drawings, but there may be times when it will be most helpful to remember that a point has no size, a line has no width, and a plane has no thickness. The plane may be the least familiar idea to you and can be thought of as a flat surface. We also agree that the word line will mean straight line. Both a line and a plane are a set of points.



It is conventional to name points with capital letters such as A or B. The line containing those points is named $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$. The arrows indicate the fact that the line extends indefinitely in both directions. For convenience, a line is sometimes named with a single lower case letter such as l or m.



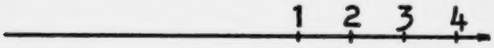
A plane is also named by a capital letter, but there's not much chance of confusing a plane with a point. A plane extends indefinitely in all directions as indicated by the top figure shown, but we find it more convenient to show it as in the bottom figure to suggest its existence and orientation in space.



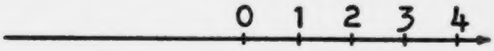
We'll get back to points, lines, and planes later. The main purpose of this dicussion is to help establish the abstract nature of the subject and the elements we work with.

Sets of Numbers

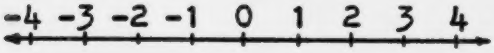
Before looking at the pure geometry aspects of our subject, we would like to be able to express the distance between any two points in space. To do this, we must talk about numbers, which takes you back to your experience with algebra. We'll review those numbers now, then later we'll review some of their basic properties. The number lines shown indicate the points associated with each set of numbers. This helps to show how each set is an extension of the previous set.



(N) Natural numbers:
Numbers used for counting



(W) Whole numbers:
Natural numbers and zero



(J) Integers: Whole numbers and their opposites

(R) Rational numbers: Numbers which can be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$

The last number line is merely an attempt to show that the points associated with the rational numbers are very closely packed and yet there are still other points on the line that are not associated with rational numbers. Since we can't write all of them on the line, some examples are as follows.

$\frac{2}{3}$, $\frac{9}{5}$, $\frac{-3}{4}$, $2 = \frac{2}{1}$, $5\frac{1}{2} = \frac{11}{2}$, $-6.7 = \frac{-67}{10}$,

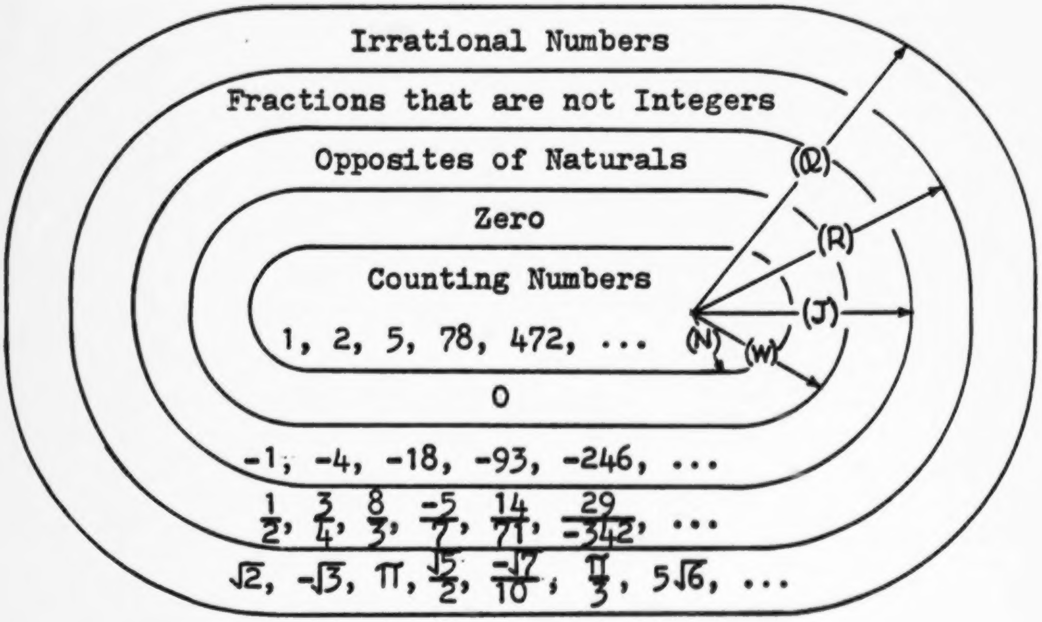
$\frac{2 + 7.3}{1\frac{1}{2}} = \frac{31}{5}$, $1 - \frac{8\frac{1}{3}}{4.1} = \frac{-127}{123}$, $2.\overline{36} = \frac{26}{11}$

To show that a number is rational, we must show that it conforms to the definition. See if your arithmetic is sharp enough to confirm that the last three examples are written correctly in the form $\frac{a}{b}$ where a and b are integers.

(ℝ) Real numbers: All numbers which can be associated with points on the number line

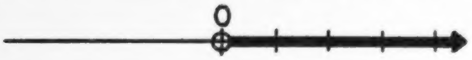
This is where we fill in the gaps in the previous line with the numbers that are not rational, that is, irrational. Examples of irrational numbers are $\sqrt{2}$, $-\sqrt{5}$, π, $\frac{\sqrt{3}}{2}$. Now the line is completely solid because there is a point for every real number and a real number for every point.

A convenient way to show the subset relations in these sets of numbers is with a Venn diagram.


Irrational Numbers
Fractions that are not Integers
Opposites of Naturals
Zero
Counting Numbers
1, 2, 5, 78, 472, ...
(Q)
(R)
(J)
(N)
(W)
0
-1, -4, -18, -93, -246, ...
$\frac{1}{2}, \frac{3}{4}, \frac{8}{3}, \frac{-5}{7}, \frac{14}{71}, \frac{29}{-342}, \ldots$
$\sqrt{2}, -\sqrt{3}, \pi, \frac{\sqrt{5}}{2}, \frac{-\sqrt{7}}{10}, \frac{\pi}{3}, 5\sqrt{6}, \ldots$

Each ring indicates the new numbers that are included together with what we already had to form the new set. Every set in the entire diagram is a subset of the set of real numbers. Complete understanding of these relationships is essential not only in geometry, but in future math courses you may take.

Now we can consider the set of numbers that we'll be particularly concerned with in geometry.



(G) Positive Real Numbers:
Numbers that may be used
as measures in geometry

This should agree with your previous knowledge of measures.

EXERCISE SET 6.1

1. What are the three undefined terms in geometry?
2. Illustrate the three undefined terms by a drawing and show how they are named.

For each example, write the letter of every number that belongs to that set.

3. Natural numbers
4. Whole numbers
5. Integers
6. Rational numbers
7. Real numbers
8. Numbers that can be used as measures in geometry

a) $-\frac{3}{4}$
b) 7
c) $\frac{15}{3}$
d) $\frac{0}{6}$
e) $\sqrt{5}$
f) -4
g) $\frac{\pi}{2}$
h) 5783
i) 3.14
j) $\frac{1}{100}$

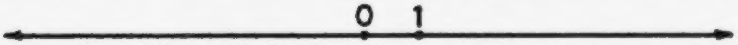
Distance

Now we must answer the question, "How do we determine what this number shall be?". Recall that we said a line may be considered a set of points - no numbers, just points.

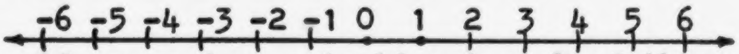
If we wish, we may select any point at random and associate it with any number we choose. This would be the only point on the line associated with a number.

We may now select any other point on the line and associate it with any other number that we choose.

But now if we select a third point, we are no longer free to associate it with any number we choose. As soon as we associate two points on a line with numbers, we have pinned down every other point on the line to a specific number. Since this is the case, we find it more practical to associate two numbers with two points as follows.

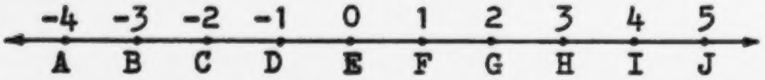


We consider the distance from 0 to 1 as our unit, and we can find the points on the line associated with other integers by using our compass to lay off that unit in both directions as far as we wish.



We could go so far as to bisect each of these segments and find the points associated with $\frac{1}{2}$, $1\frac{1}{2}$, $-\frac{1}{2}$, etc., and find as many pairs of points and numbers as we like.

We have established a coordinate system on a number line. Each number is called the coordinate of the point associated with it. It is customary to name points with capital letters as shown below.



So the coordinate of B is -3, the coordinate of F is 1, etc. For this set of points, we can determine the distance between any two of them by simply counting units.

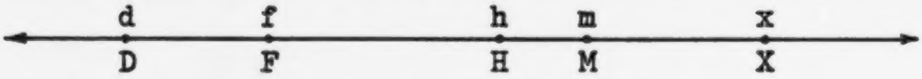
The distance from G to J is 3.
The distance from F to B is 4.
The distance from A to H is 7.

But it would be rather awkward to have to find distance in this way for all pairs of points. Using our super powers of observation, we see that the distance from G to J is also the difference of the coordinates, $5-2=3$. If we find the difference of the coordinates of F and B, we might write $(-3)-(1)=-4$. But we know distance must be positive, so we would have to write instead, $(1)-(-3)=4$.

This brings us to a discussion of the order of the real numbers. For any two real numbers, the number whose associated point is on the right is the larger number. We know from arithmetic that when subtracting a smaller number from a larger number the result is always positive. This is true for all pairs of real numbers. Since -3 is smaller than 1, $(-3<1)$, this is why we must write $(1)-(-3)$ rather than $(-3)-(1)$. The coordinates of A and H are -4 and 3. Since -4 is smaller than 3, we write the difference as $(3)-(-4)=7$.

So the distance from D to B is $(-1)-(-3)=2$, and the distance from A to E is $(0)-(-4)=4$.

We would like to generalize this whole idea of distance. When we generalize numbers and points on a line, it is conventional to use capital letters for the points and the same lower case letter for the coordinate of that point.



From our previous examples we would write

The distance from F to M is $m-f$.
The distance from X to H is $x-h$.

Without a number line to look at, we can write the following general statement.

> For any two points, A and B, of a line, with coordinates a and b respectively, the distance from A to B is $a - b$ if $b < a$, or $b - a$ if $a < b$. It is denoted by AB or BA.

The First Postulates

In all of the previous discussion, we have established two very important ideas.

1. The distance between two points in space is a positive real number.
2. That number is determined by taking the positive difference of the coordinates of the points on the line containing the points.

But we didn't actually prove anything. We established them as being reasonable in view of what we feel distance should be and how it should be determined. When we make statements in this spirit, as we will many times in geometry, we call the statements postulates. Together with definitions, they become the foundation stones of our subject.

Therefore, we now formally make these statements as our first postulates and define the word distance. (The word unique in mathematics means exactly one.)

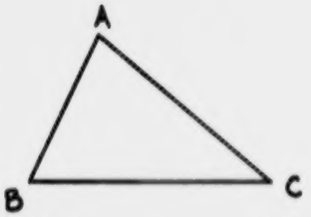
Postulate 1: For every pair of different points in space there is an associated unique positive real number. (The Distance Postulate)

Definition: The measure of the distance between two points is the number given by Postulate 1. If the points are R and S, the distance is denoted by RS or SR.

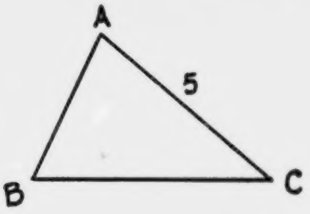
Postulate 2: The points of a line can be placed in correspondence with the real numbers in such a way that
1) Any two points may be paired with 0 and 1.
2) For every point there corresponds exactly one real number.
3) For every real number there corresponds exactly one point.
4) The distance between any two points is the positive difference of the numbers corresponding to those points.
(The Ruler Postulate)

In effect, Postulate 2 is providing us with an "ideal" ruler which extends indefinitely in both directions from 0. The significance of all this can be discussed as follows.

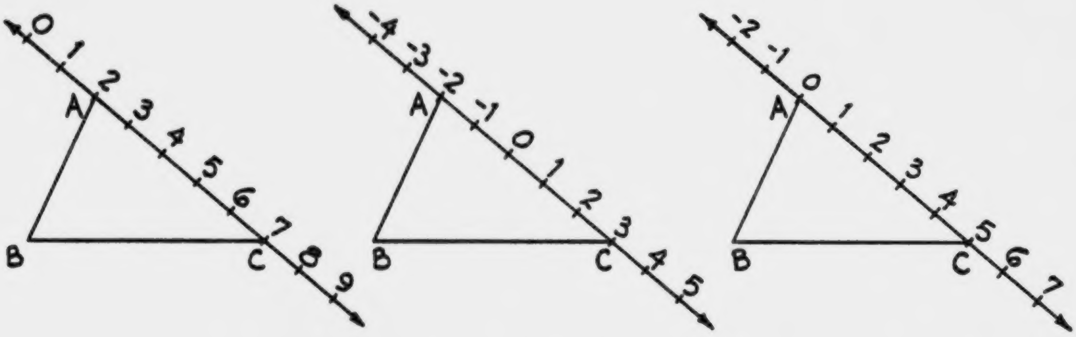
We will work with many figures during the course such as a triangle.



The Distance Postulate tells us that the distance from A to C is a positive real number. Let's call it 5.



The Ruler Postulate tells us that the points of the line containing A and C may be set into one to one correspondence with the real numbers such that any of the following would hold.



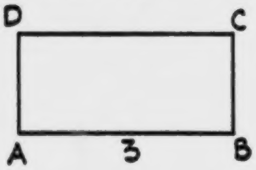
Of course, being the bright souls that we are, we recognize that the figure on the right illustrates what we would do in measuring the distance from A to C with our "physical" ruler, except that there wouldn't be anything to the left of 0, and the 0 itself would not be written.

Fortunately, we won't have to go through all this every time we want to say that the distance between two points is 14, but we'll be secure in the knowledge that there was something back there that allowed us to say it.

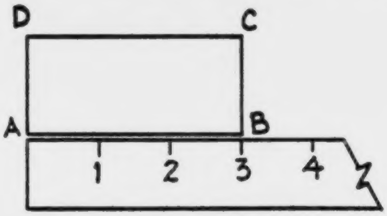
Measure and Measurement

This is a good time to discuss the sometimes troublesome use of the words measure and measurement. We'll consider them now with respect to distance, but the same distinctions will be made when we consider angles, area, etc.

The word measure is used both as a noun and as a verb. As a noun, it is the number as referred to in the distance postulate, and as determined by the ruler postulate, and it is *exact*. If we *originate* a problem involving the figure shown, we may say that the measure of the distance from A to B is exactly 3. That is, a coordinate system could be established which could pair A with 0 and B with 3.



When the word measure is used as a verb, it refers to the process of trying to find the distance between two given points. The number arrived at by this process is called a *measurement*, and it is *approximate*. In this case we use a coordinate system with a pre-established unit and apply it to the given points. If the unit selected is a centimeter, then we may say that the measurement is 3 centimeters, but this is *never exact*. We could pair the point A with 0 alright, but since the given figure and the pre-established unit are independent of each other, there is no way we can say that B does actually correspond to 3. Remember that a point has no size, so even with your special superman contact lenses, you cannot "physically see" point B. Therefore, there is no way that you can distinguish point B from the point that is $\frac{1}{1,000,000}$ centimeter to the left of it or to the right of it. In fact, the "physical point" that you do see in the width of the printed line contains all three of these points and infinitely many more.

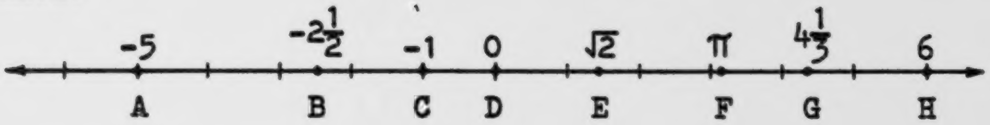


This is no problem in practical applications because we can find a measurement as close, or *precise*, as we need by selecting a unit as small as we wish for the purposes of that problem. Sometimes the word measurement is used to describe the process itself, but this should cause no confusion. A complete discussion of measurements would require an entire chapter and would be worthwhile. You may find it interesting and beneficial to look into it further in a good reference book.

The distinction between these two words, measure and measurement, further emphasizes the abstract nature of our approach to geometry. Your ability to make this distinction here will aid you in grasping many concepts that will confront you later. If you're not sure whether you understand it, read it again, and read it slowly, and really concentrate.

EXERCISE SET 6.2

For each example write the distance between the points named.



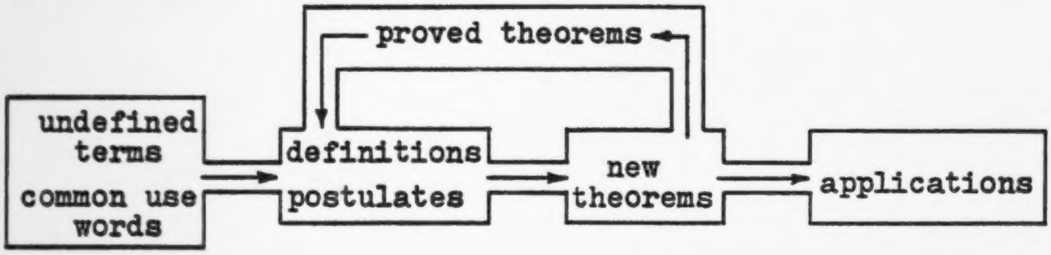
1. AC
2. BD
3. EF
4. HD
5. GB
6. BH
7. EA
8. CF
9. GF
10. HA
11. The distance between two given points is measured with a ruler and recorded as 5 inches. Is this exact or approximate?
12. Two points on a ruler have the numbers 2 and 6 written beside them. If the distance between them is stated as 4, is this exact or approximate?

CHAPTER 7 - THE STRUCTURE OF GEOMETRY
or STARTING FROM SCRATCH

Basic Structure

Of course, we've already begun to develop this structure, but this is a good time to develop an overview of the entire process. We'll try to show how it all ties together so that you will know what your general goals are as you move through the course.

The diagram shown here illustrates generally how any deductive system operates. You've been through this when you took algebra, although the process is not always emphasized in that subject.



In geometry, we've stated that our undefined terms are point, line, and plane. We will use these together with our common usage words to make definitions and state postulates. The nature of the course is actually determined more by the postulates than anything else. We've stated two postulates already, related to distance, but some further comments are in order. A postulate is a basic assumption of a statement that seems so reasonable that we can accept it without proof. When Euclid started putting it together, he observed the physical world in which he lived to base his postulates on. Naturally, he observed a "flat" world. His postulates would seem reasonable to us because we observe that same "flat" world, even though we know that we live on a sphere 8000 miles in diameter. But what if we lived on a sphere 8 miles in diameter? Is it possible that the observations we would make then about "space" would be different from the ones we now make? The point is that the physical world we observe becomes the "model" for our assumptions about space, that is, our postulates. But other models are possible and, in fact, have been explored with very significant consequences.

Once postulates are accepted, these, together with definitions, are used to prove other statements which we call theorems. In the first stages of the course you may find that some of the theorems appear to be just as acceptable without proof as the postulates. However, we try to accept only the minimum number of postulates that we need to do the job. Although you will find many theorems later in the course which are definitely not obvious, one of the basic differences between a postulate and a theorem is that a theorem can be proved. In fact, if we should accept a statement as a postulate now, and then discover later that it could be proved from the other postulates, we would have to call it a theorem instead just for that reason.

Notice in the diagram that the second box contains only definitions and postulates initially. When the first theorem is proved, these become the reasons, or authorities, for the statements in the proof. But just as soon as this theorem, or any theorem, is proved, it now goes back into the box with definitions and postulates to be available as another reason to help prove new theorems. Herein lies the "power" of the course. From a relatively small list of basic assumptions, or postulates, we build a very large superstructure of knowledge by using what we've already proved and the deductive reasoning process.

As indicated in the diagram, a new theorem may have direct application. But it's also possible that its main purpose is to help prove other theorems. You must remember throughout the course that any statement you make in a proof must have a reason and that reason must be a definition, a postulate, or a proved theorem already established in this course. Some statements are considered as given and require no other reason.

Definitions

Definitions play a very vital role in this whole process and we will make a lot of them. You will become familiar with them partly through use and partly through memorization, but this is a task that you must accept, as no one else can do it for you.

Definitions have important characteristics that we must consider both in writing them and in using them. When stated as a simple sentence, the subject is the thing being defined, and the predicate gives the defining characteristics.

Example:

A bolo is a large single-edged knife used by the natives of the Phillipines.

In a good definition, we should be able to interchange the subject and the predicate and retain the meaning of the sentence.

A large single-edged knife used by natives of the Phillipines is a bolo.

An example of a statement that is not a definition was given earlier about a line.

A line is a set of points.

This is true and it does tell us something about a line, but if we interchange the subject and predicate, we obtain a statement which is not meaningful.

A set of points is a line.

This is of no help since there are many sets of points which are not lines. Therefore, this is not an acceptable definition.

Many definitions are given as conditional statements. In this case, both the statement and its converse must be true.

If two rivers flow together, then they form a confluence.
If two rivers form a confluence, then they flow together.

Earlier we found it convenient to write such statements as a single biconditional statement.

Two rivers form a confluence if and only if they flow together.

This is a particularly important characteristic since you will use definitions many times as reasons for statements in a proof. Depending on the nature of the proof, you may state a reason in one form in one situation and in the converse form in a different situation. Just remember that when a definition is given, you have two possible reasons available when working with proof. This point will be reemphasized and clarified when it actually arises.

EXERCISE SET 7.1

1. A basic assumption that is accepted without proof is a __________.
2. A proved statement is a __________.
3. Name the three kinds of reasons that may be given for statements in a proof.

For each of the following, rewrite the statement interchanging the subject and predicate. Tell whether or not this is an acceptable definition for the underlined word.

4. A cowboy is a man that is usually around cows.
5. A canyon is a narrow valley with high steep sides.
6. A bank of earth or a dam built as a defense against flooding by a river or the sea is a dike.
7. A violin is a musical instrument played with a bow.

For each of the following, write the biconditional of the statement and its converse. Tell whether or not this is an acceptable definition for the underlined word.

8. An item is obsolete if it is no longer in use.
9. If an animal is a flesh eater, then it is carnivorous.
10. If a number is divisible by 4, then it is an even number.
11. A triangle is isosceles if it has at least two sides of equal length.

Some Basic Definitions

These are the first of many definitions that will be made throughout the course as they are needed.

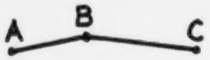
Space is the set of all points.

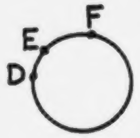
A set of points is collinear if and only if they all lie on the same line.

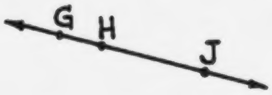
A set of points is coplanar if and only if they all lie in the same plane.

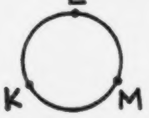
We need to discuss the word "between" before making the next definition. Euclid failed to consider this relation and it was later shown to have resulted in some logical weaknesses in his work.

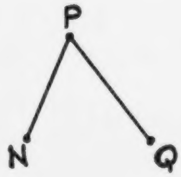
In each of the following figures, would you consider a particular point to be between the other two?





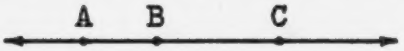






You might consider B and E as possibilities in the first two figures, but a look at the last two figures suggests that we would have problems with a definition for similar cases. We find the third figure to be the most reasonable in which to state a betweenness relation. You surely said H is between G and J. To use this as a basis for a definition, we notice that there are two conditions which must be met. We state this formally in the following manner.

Definition of betweenness of points: Point B lies between points A and C if and only if (1) A, B, and C are collinear, and (2) $AB + BC = AC$.



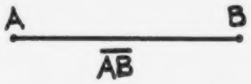
Remember that AB is the designation for the measure of the distance from A to B. The first condition requires that all three points lie on one line and the second condition identifies which point is between the other two.

If we are not given a figure but are told that points R, S, and T are collinear and $SR + RT = ST$, we know that R is between S and T.

The equation in condition (2) of the definition is the type of statement you will make frequently in proofs. The reason you will give for it in each case is "Definition of betweenness of points".

A *line segment* is the union of two points in space and all points between them. We will call it simply a segment.

If A and B are two points in space, $\overline{AB}$ names the segment determined by these two points. A and B are the *endpoints* of the segment.

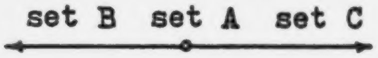


The *length of a segment* is the distance between its endpoints.

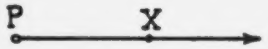
For $\overline{AB}$, we have indicated the distance from A to B as AB. We will now use AB as the length of $\overline{AB}$. Make sure you distinguish between the two symbols.

$\overline{AB}$ is a segment and is a set of points.
AB is a length and is a number.

A designated point of a line separates the line into three non-intersecting sets; set A (the designated point), and sets B and C such that if a point from B and a point from C are the endpoints of a segment, that segment will contain the designated point. Each of the sets B and C is called a half-line. The designated point is the vertex of the half-line. The vertex does not belong to the half-line.

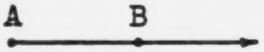


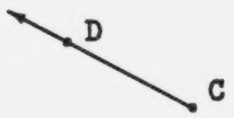
P is the vertex of the half-line containing X. It is shown with an open circle to indicate that it is not part of the half-line.



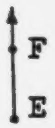
A ray is the union of a half-line and its vertex.

The ray shown is named $\overrightarrow{AB}$ and A is the endpoint of the ray. The endpoint of the ray is always named first and the second letter may be any other point of the ray. The direction of the ray has no effect on the direction of the arrow in the name of the ray.

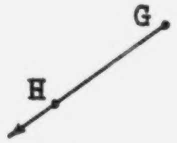




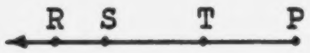
$\overrightarrow{CD}$



$\overrightarrow{EF}$



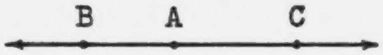
$\overrightarrow{GH}$



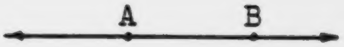
$\overrightarrow{PR}$ or $\overrightarrow{PS}$ or $\overrightarrow{PT}$

Two rays are opposite rays if and only if they are collinear and their intersection is the endpoint of each ray.

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays.



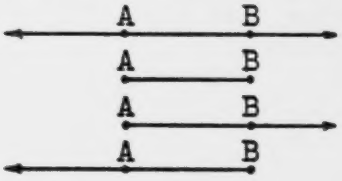
Even though the figure shown here is a line, we can name any of the following, meaning we are naming a particular set of points in the figure.



$\overleftrightarrow{AB}$ $\overline{AB}$ $\overrightarrow{AB}$ $\overrightarrow{BA}$

In summary, each of the above would name the figures below.

$\overleftrightarrow{AB}$
$\overline{AB}$
$\overrightarrow{AB}$
$\overrightarrow{BA}$



Two segments are congruent if and only if they have the same length.

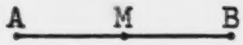
The symbol $\cong$ is read "is congruent to". If $AB = CD$, then we may write $AB = CD \longleftrightarrow \overline{AB} \cong \overline{CD}$. Remember that this may be written as two statements,

$$AB = CD \longrightarrow \overline{AB} \cong \overline{CD} \quad \text{and} \quad \overline{AB} \cong \overline{CD} \longrightarrow AB = CD$$

Notice that the = sign is used in statements about numbers, and the $\cong$ sign is used in statements about sets of points. This is an important distinction that you must make throughout the course, not only with segments and lengths, but with other types of figures and their measures as well.

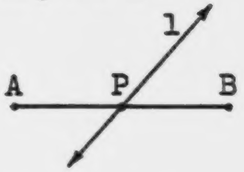
A point is the midpoint of a segment if and only if it divides the segment into two segments of equal length.

M is the midpoint of $AB \longleftrightarrow AM = MB$



A line is a bisector of a segment if and only if it intersects the segment at its midpoint.

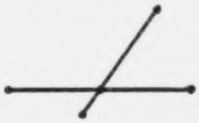
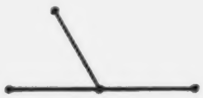
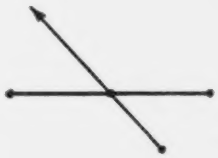
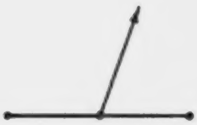
l bisects $\overline{AB} \longleftrightarrow P$ is the midpoint of $\overline{AB}$



We would normally write this as follows.

l bisects $\overline{AB} \longleftrightarrow AP = PB$

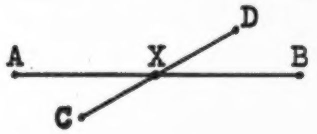
A segment or a ray may also be a bisector as illustrated in the following figures.

Notice in the first two figures that one segment may bisect a second segment and the second segment not bisect the first. If two segments bisect each other, we would have

$\overline{AB}$ bisects $\overline{CD} \longleftrightarrow CX = XD$

$\overline{CD}$ bisects $\overline{AB} \longleftrightarrow AX = XB$



These should hold you for a while. If you run into the need for one of these definitions later on and you're not quite sure of it, don't hesitate to turn back to this section and review it.

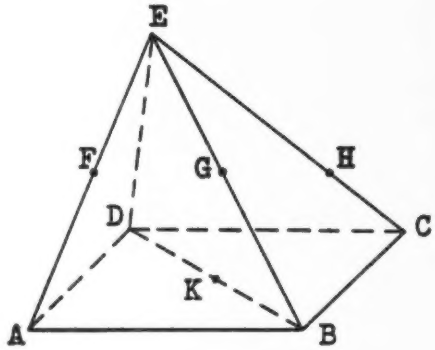
EXERCISE SET 7.2

In the space figure shown, tell whether or not the points named are collinear.

1. A, F, E
2. A, B, C
3. F, G, H
4. B, D, K
5. E, H, C
6. E, G, B, K

Tell whether or not the points named are coplanar.

7. A, B, E
8. B, C, E, F
9. G, D, K
10. A, B, C, K

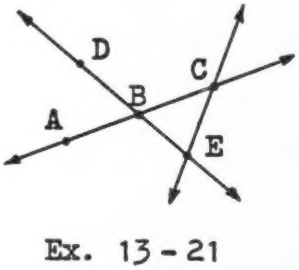


Ex. 1 - 10

11. If X, Y, and Z are three collinear points, write the statement that requires Z to be between X and Y.
12. If L, M, and N are three collinear points, and LM + NL = MN, which point is between the other two?

For the figure shown, write the name of each of the following using proper notation.

13. The line containing A and C.
14. Two different lines containing E
15. The segment with endpoints B and D
16. The ray with endpoint E and containing C
17. Two opposite rays with endpoint B



Ex. 13 - 21

Using the previous figure, complete each implication.

18. B is the midpoint of $\overline{AC} \longrightarrow$

19. B is between A and C $\longrightarrow$

20. $BC = EC \longrightarrow$

21. $\overline{AB} \cong \overline{DB} \longrightarrow$

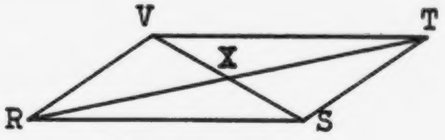
Complete each of the following biconditionals.

22. $\overline{RV} \cong \overline{ST} \longleftrightarrow$

23. $RS = VT \longleftrightarrow$

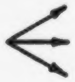
24. X is the midpoint of $\overline{VS} \longleftrightarrow$

25. $\overline{VS}$ bisects $\overline{RT} \longleftrightarrow$



Ex. 22 - 26

26. Write three different statements that could be implied as shown.

$XV = XS$ 

27. If two segments bisect each other, must all four segments be congruent?

28. Can a line be bisected? Can a ray be bisected?

Let A, B, and C be points of a number line and B is the midpoint of $\overline{AC}$.

29. If the coordinate of A is 3 and C is 10, what is the coordinate of B?

30. If the coordinate of B is 4 and C is 15, what is the coordinate of A?

Postulates about Points, Lines, and Planes

The first postulates of geometry are not very exciting and may even seem trivial to you. The important thing is that they seem reasonable and that they do appear to describe the physical world as we see it.

Before introducing the postulates of this section, we need to discuss the meanings of two words, existence and uniqueness.

Existence states that the thing being considered does actually exist. It generally means there is "at least one".

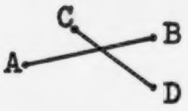
Uniqueness states that there is "not more than one" of the thing being considered.

Examples:

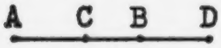
Consider the numbers that make each statement true.

$x + 2 = 3$	We have existence and uniqueness.
$x + 2 > 3$	We have existence but not uniqueness.
$x + 2 = x + 3$	We do not have existence.

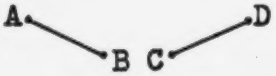
Consider the points that $\overline{AB}$ and $\overline{CD}$ have in common.



Existence and uniqueness



Existence but not uniqueness



Not existence

When both existence and uniqueness occur as in the first case of each example, we say that there is one and only one, or exactly one.

There are times when it is necessary to establish existence and uniqueness in geometry. After all, if you're going to talk about something, you would like to be assured that it does exist, wouldn't you? Or you may wish to be certain that there is just exactly one of these things.

Remember that we've already stated Postulate 1 (The distance postulate) and Postulate 2 (The ruler postulate). All postulates will be numbered for reference, but you do not need to learn them by number.

Postulate 3 is purely an existence postulate and has three cases.

Postulate 3: a) Every line contains at least two points.
b) Every plane contains at least three noncollinear points.
c) Space contains at least four noncoplanar points.

Postulates 4 and 5 are existence and uniqueness postulates.

Postulate 4: For any two different points, there is exactly one line containing both points.

Postulate 5: For any three noncollinear points, there is exactly one plane containing the three points.

Whenever there is exactly one of something, it is said to be determined. We can restate postulates 4 and 5 as follows.

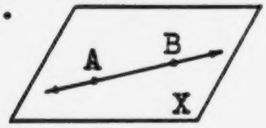
Postulate 4: Two different points determine a line.
Postulate 5: Three noncollinear points determine a plane.

Postulate 4 should give you no problem, but if postulate 5 is at all vague, grab a friend and try this little experiment. Find three pencils of different length and stand them upright on your desk. Now place a flat surface, book, notebook, etc., so that it contacts all three points. Do you see that no matter how the points are placed, there is always exactly one position for the plane? Would this be true for two points? for four points?

Postulates 6 and 7 state relationships involving lines and planes. your acceptance of them requires your ability to visualize relations in space.

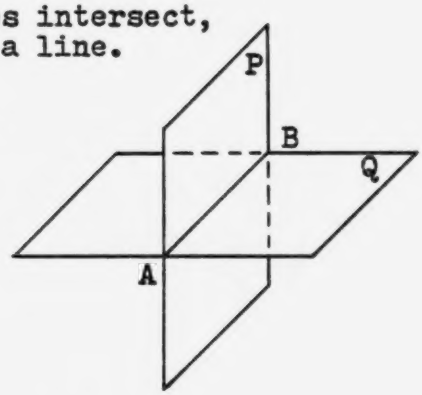
Postulate 6: If two points of a line are in a plane, then the line is in that plane.

A and B are in X $\longrightarrow \overleftrightarrow{AB}$ is in X



Postulate 7: If two different planes intersect, their intersection is a line.

P and Q intersect $\longrightarrow P \cap Q = \overleftrightarrow{AB}$



Our First Theorems

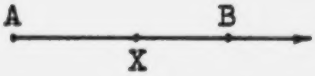
At this stage we can begin to state some very basic theorems. These first theorems may seem to be just as acceptable to you without proof as the postulates. Remember that a statement may be a theorem simply because it can be proved.

All these theorems are existence and uniqueness theorem. That is, they all contain the phrase "exactly one".

All theorems in the book will be designated by the chapter number and theorem number within that chapter. Once again, you do not have to learn them by number.

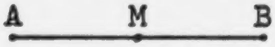
The proof of the first theorem involves the use of the Ruler Postulate by establishing a coordinate system.

Theorem 7-1: On a ray, there is exactly one point at a given distance from the endpoint of the ray. (Point plotting theorem)



Let $\overrightarrow{AB}$ be the given ray and let x be the given distance, which is a positive number. We may associate A with 0 and all other points of AB with positive numbers. Since there is exactly one point X, associated with the real number x, that is the desired point.

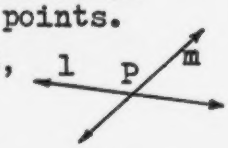
Theorem 7-2: A segment has exactly one midpoint.



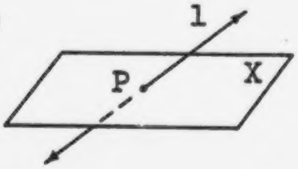
Let $\overline{AB}$ be the given segment. A midpoint must satisfy two conditions. Since it is between A and B, then $AM + MB = AB$ by the definition of betweenness of points. Also $AM = MB$ by the definition of a midpoint. These two equations can be shown by algebra to imply $AM = \frac{1}{2}AB$. Since $\frac{1}{2}AB$ is a distance, the previous theorem states that there is exactly one point on $\overrightarrow{AB}$ at that distance from A. Therefore, the segment has exactly one midpoint.

These are informal arguments in paragraph form and it is not necessary to dwell on them, as this is not the type of proof we will generally be concerned with. It does illustrate how one theorem is used to help justify the next one. The theorems will be used later when it is necessary to justify the uniqueness of these points.

Theorem 7-3: If two different lines intersect, they intersect in exactly one point.

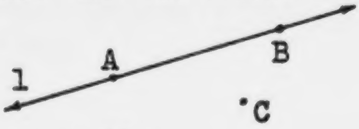


Theorem 7-4: If a line intersects a plane not containing it, the intersection is exactly one point.



It may seem strange to call these theorems, but they can be proved. However, the proof is rather awkward at this point in the course, and we'll not consider them here.

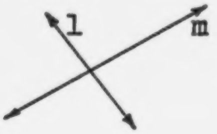
Theorem 7-5: If a point is not on a line, exactly one plane contains the point and the line.



We know that there are at least two points on l by postulate 3. Call them A and B. Since C is not on l, A, B, and C are noncollinear. By postulate 5 there is exactly one plane containing A, B, and C. We also know that l must lie in that plane by postulate 6.

This is another paragraph proof which you needn't spend much time on. Notice that it is another way in which a plane is determined. The proof of theorem 7-6 would be similar, and we will state the theorem without proof.

Theorem 7-6: If two lines intersect, exactly one plane contains both lines.



EXERCISE SET 7.3

State whether each of the following illustrates existence and uniqueness, existence but not uniqueness, or no existence.

1. Faces on Mt. Rushmore
2. Stars in the Solar System
3. Diagonals of a triangle
4. Solutions of the equation $x^2 = 4$
5. Midpoints of a given segment

Write the postulate that justifies each of the following.

6. R, S, and T are three noncollinear points. Therefore, there is exactly one plane containing R, S, and T.
7. P and Q are intersecting planes. Therefore, the intersection of P and Q is a line.
8. X is a plane. Therefore, X contains at least three noncollinear points.
9. C and D are two different points in plane Z. Therefore, $\overleftrightarrow{CD}$ lies in plane Z.
10. M and N are two different points. Therefore, $\overleftrightarrow{MN}$ is unique.
11. l is a line. Therefore, l contains at least two points.

Write the theorem that justifies each of the following.

12. Lines k and j intersect at point A.
Therefore, A is the only point of intersection of k and j.

13. F is a point not on line n.
Therefore, exactly one plane contains F and n.

14. $\overrightarrow{KL}$ is a given ray and d is a given distance.
Therefore, there is exactly one point X on $\overrightarrow{KL}$ such that $KX = d$.

15. a and b are intersecting lines.
Therefore, a and b determine a plane.

16. Line m intersects plane P at point R. P does not contain m. Therefore, R is the only point of intersection of m and P.

17. B is a midpoint of $\overline{EF}$.
Therefore, B is the only midpoint of $\overline{EF}$.

18. Name three ways in which a plane is determined.

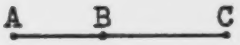
Algebra Properties Used in Geometry

Since you enjoyed algebra so much, we don't want to leave out the essentials of that course that you will need in geometry. Some of these properties are postulates and some are theorems, but for our purposes it is more convenient to use the word "property" for all of them. We'll not bother to prove any that are theorems anyway.

We'll state five of the basic properties of real numbers, but these are not frequently used in geometry.

Commutative property of addition	$a+b = b+a$
Commutative property of multiplication	$a \cdot b = b \cdot a$
Associative property of addition	$(a+b)+c = a+(b+c)$
Associative property of multiplication	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Distributive property	$a \cdot (b+c) = a \cdot b + a \cdot c$

The first four of these are properties that you will be able to use without a separate step.
For example, in the figure shown, you may write the equation $AB + BC = AC$.



If at some other point in the same problem you wish to consider it as $BC + AB = AC$ or $CB + BA = CA$, you may do so without comment.

The algebra properties used most frequently in geometry are the properties of equality. We'll state them now along with the properties of inequality which are important in a particular unit. The addition and subtraction properties of equality list several consequences. Any of these may be applied in a single step in a proof with the same name used such as "addition property of equality".

Properties of Equality

Reflexive property:	$a = a$
Symmetric property:	If $a = b$, then $b = a$
Transitive property:	If $a = b$ and $b = c$, then $a = c$
Addition property:	If $a = b$, then $a + c = b + c$
Consequences	If $a = b$ and $c = d$, then $a + c = b + d$ If $a - b = c$, then $a = c + b$ If $a - c = b - c$, then $a = b$
Subtraction property:	If $a = b$, then $a - c = b - c$
Consequences	If $a = b$ and $c = d$, then $a - c = b - d$ If $a + b = c$, then $a = c - b$ If $a + c = b + c$, then $a = b$
Multiplication property:	If $a = b$, then $ac = bc$
Division property:	If $a = b$, then $\frac{a}{c} = \frac{b}{c}$, $c \neq 0$
Substitution property:	If $a = b$, then either name may be used in place of the other in an equation or inequality.
Zero-product property:	If $a \cdot b = 0$, then $a = 0$ or $b = 0$

The next statement is not a single property, but it is one which we apply in a single step just as you did in algebra. We'll show the justification first, and then use the name listed as the reason for applying it in a single step.

$a + a = 1a + 1a$	Multiplication property of 1
$= (1 + 1)a$	Distributive property
$= 2a$	Addition fact

We will use the name that you used in algebra.

Combining like terms: $a + a = 2a$

The properties of inequality listed are just those that you are likely to need for this course.

Properties of Inequality

Definition of $<$: If $a = b + c$ and c is positive, then $b < a$
Definition of $>$: If $b < a$, then $a > b$

Transitive property: If $a < b$ and $b < c$, then $a < c$

Addition property: If $a < b$, then $a + c < b + c$

Multiplication properties: If $a < b$ and $c > 0$, then $ac < bc$
If $a < b$ and $c < 0$, then $ac > bc$

Trichotomy: If a and b are real numbers, exactly one of the following is true. $a < b$, $a = b$, $a > b$

When these algegra properties are applied in a geometry situation, naturally they are not going to look like they do here. For example, the addition property of equality states "If $a = b$, then $a + c = b + c$". But if a, b, and c are lengths of segments, the application may look something like this.

$$AB = BC \longrightarrow AB + CD = BC + CD$$

The reason given for this implication would be the addition property of equality where $a = (AB)$, $b = (BC)$, and $c = (CD)$.

EXERCISE SET 7.4

Write the name of the property illustrated

1. $(AB + BC) + CD = AB + (BC + CD)$
2. $3(x + 8) = 3x + 24$
3. $RS \cdot ST = ST \cdot RS$
4. $(BC + AB) + CD = (AB + BC) + CD$
5. $x \cdot (x \cdot 5) = (x \cdot x) \cdot 5$
6. $(x + 2) \cdot (x + 3) = (x + 2) \cdot x + (x + 2) \cdot 3$
7. $8 + 3y = 3y + 8$

Write the name of the property of equality illustrated.

8. $AB = CD \longrightarrow AB + BC = CD + BC$

9. $XY = AB \longrightarrow AB = XY$

10. $RS + RS = 2 \cdot RS$

11. $(x - 6)(x + 4) = 0 \longrightarrow (x - 6 = 0 \text{ or } x + 4 = 0)$

12. $AB = AB$

13. $EH - FH = EF \longrightarrow EH = EF + FH$

14. $\left.\begin{array}{l} RS = ST \\ ST = TV \end{array}\right\} \longrightarrow RS = TV$

15. $MN + NP = PR + NP \longrightarrow MN = PR$

16. $\left.\begin{array}{r} AB + BC = AC \\ BC = XY \end{array}\right\} \longrightarrow AB + XY = AC$

17. $2 \cdot AB = 2 \cdot CD \longrightarrow AB = CD$

18. $\left.\begin{array}{l} AB = RS \\ BC = ST \end{array}\right\} \longrightarrow AB + BC = RS + ST$

19. $CD + DE = CE \longrightarrow CD = CE - DE$

20. $\left.\begin{array}{r} AB + BC = CD + BC \\ AB + BC = AC \\ CD + BC = BD \end{array}\right\} \longrightarrow AC = BD$

Write the name of the property of inequality illustrated.

21. $2 \cdot AB < 2 \cdot CD \longrightarrow AB < CD$

22. $\left.\begin{array}{l} AB < CD \\ CD < EF \end{array}\right\} \longrightarrow AB < EF$

23. $AC = AB + BC \longrightarrow AB < AC$

24. AB and CD are measures $\longrightarrow (AB < CD \text{ or } AB = CD \text{ or } AB > CD)$

25. $3 < 4 \longrightarrow -6 > -8$

26. $RS < XY \longrightarrow RS + ST < XY + ST$

27. $XY < YZ \longrightarrow YZ > XY$

CHAPTER 8 - FLOW PROOF

or AWAY WE GO →

There are different ways of organizing the statements in a proof. We've already illustrated the paragraph form in some of the first theorems. An organization that has been commonly used in geometry is called the two-column form. The statements in the proof are made in one column and the reasons for these statements are made directly opposite them in the second column.

This book is going to employ an organization called the flow proof form. This will make use of the symbolism for the implication, $p \rightarrow q$, and should show clearly how the deductive reasoning process is involved in a proof. The statements will flow in a logically connected sequence throughout the proof. We'll first discuss a very important principle of logic which, together with the Law of Detachment, provides the real power of deduction.

The Chain Rule

Suppose the following statements were both true.

If the Hornets win tonight, then they will be state champs.

If the Hornets are state champs, then Monday will be a holiday.

Wouldn't you agree that the following statement would also have to be true.

If the Hornets win tonight, then Monday will be a holiday.

This illustrates the Chain Rule in logic, which can be stated symbolically as follows.

If $p \rightarrow q$ and $q \rightarrow r$, then $p \rightarrow r$

Later and more significant proofs will involve the stringing together of whole bunches of statements in which the ultimate valid conclusion may not be at all obvious, and which may appear to be far removed from the initial statements in the proof. This is the real strength and ultimate goal of our course.

Flow Proof

The first examples we will now consider will admittedly be elementary and the conclusions not very startling. But this is for the best, as our purpose here is to become familiar with a method, and to see what actually constitutes a proof. Our first proofs will rely heavily on the properties of algebra.

We'll look first at a relationship that was suggested in one of the earlier paragraph proofs but not actually proved.

If M is the midpoint of $\overline{AB}$, then $AM = \frac{1}{2}AB$

We can identify the hypothesis and conclusion as follows.

Hypothesis: M is the midpoint of $\overline{AB}$

Conclusion: $AM = \frac{1}{2}AB$

When writing the proof, we consider the hypothesis as that which is "given", and the conclusion as that which we are to "prove". Generally a figure is drawn or constructed representing the conditions of the hypothesis. Some relationships are acceptable on the basis of a figure and some are not. In the figure shown here, the betweenness relationship is acceptable from the figure, but the midpoint relation is not, and must be stated in the hypothesis. The statements and reasons for the statements are made with corresponding numbers, and together they constitute the argument for the proof. We emphasize the chain rule at the end of the statements by indicating that the hypothesis does therefore imply the conclusion. The symbol for the word "therefore" is ∴. The following is the form that you will use for proofs throughout the course.

Given: M is the midpoint of $\overline{AB}$

Prove: $AM = \frac{1}{2}AB$

Argument

$$\left.\begin{array}{l} \textcircled{1}\ \text{M is the midpoint of } \overline{AB} \xrightarrow{\textcircled{2}} AM = MB \\ \textcircled{3}\ \text{M is between A and B} \xrightarrow{\textcircled{4}} AM + MB = AB \end{array}\right\} \xrightarrow{\textcircled{5}} AM + AM = AB \xrightarrow{\textcircled{6}}$$

$$\xrightarrow{\textcircled{6}} 2\,AM = AB \xrightarrow{\textcircled{7}} AM = \frac{1}{2}AB$$

$$\therefore \text{M is the midpoint of } \overline{AB} \longrightarrow AM = \frac{1}{2}AB$$

① Given
② Definition of a midpoint
③ Acceptable from the figure (Normally statement ③ will not be written in a proof.)
④ Definition of betweenness of points
⑤ Substitution (AM from ② for MB in ④)
⑥ Combining like terms
⑦ Multiplication property of equality (Multiply both sides of ⑥ by 1/2)

Read the following comments and then restudy the proof.

When you approach the right edge of your paper, be sure to break the proof at an arrow. Leave a slightly wider horizontal space and then repeat that arrow with the same number back at the left edge.

We consider the number of a statement to be the number to its left. But when that number is placed above an arrow, it emphasizes the fact that the reason involves both the statement to the left and to the right of the arrow.

When two or more statements imply a single statement, they are placed in braces as shown with statements ② and ④. In the case of substitution, as this one is, two or more statements are always required to imply another statement. To the left of the arrow, you must write the statement or statements being substituted from and the statement being substituted into. To the right of the arrow will be the statement being substituted into after the substitution has been made.

We've emphasized the importance of the Law of Detachment in proof so we'll discuss it here just for statements ① and ②.

If a general conditional statement, p→q, is true and the hypothesis, p, detached for a specific case, is true, then the conclusion, q, for that specific case, is true.

The general conditional in our example is reason ②, the definition of a midpoint. We would state it here as,

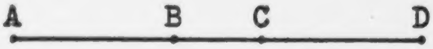
If a point is the midpoint of a segment, then it divides it into two segments of equal length.

We already know this to be true since it is a definition. The statement p, "a point is the midpoint of a segment", detached for our specific case is "M is the midpoint of $\overline{AB}$". We also know this to be true since it is given. By the Law of Detachment, the conclusion q, "it divides it into two segments of equal length", specifically "$AM = MB$", must also be true.

The general conditional referred to in the Law of Detachment will always be a reason in a proof. It must be available first even though it may be written last. The statements in the argument to the left and right of the arrow with the same number as the reason will be the specific, detached, cases for that reason. This will be true of every implication in every proof although it is sometimes difficult to recognize.

We'll present one more example for study. Sometimes reasons may be abbreviated or symbolized to save writing. (Aren't we nice!) But you must earn the right to use an abbreviation by making sure you know the complete formal statement for the reason before shortening it or symbolizing it in any way.

Given: $AB = CD$
Prove: $AC = BD$



Argument

$$① \; AB = CD \xrightarrow{②} \left.\begin{array}{r} AB + BC = CD + BC \\ ③ \; AB + BC = AC \\ ④ \; BC + CD = BD \end{array}\right\} \xrightarrow{⑤} AC = BD$$

$\therefore AB = CD \longrightarrow AC = BD$

① Given
② Add. prop. of =
③ Def. of bet. of pts.
④ Same as ③
⑤ Substitution (③ and ④ into ②)

The substitution step here involves two independent substitutions into a single statement. The right member of statement ③ is substituted into the left member of statement ② and the right member of statement ④ is substituted into the right member of ②. (Commutative property of addition is applied without stating it.) Statement ⑤ is actually a repeat of statement ② after the substitutions are made. For some reason, substitution has been a very troublesome property to apply for many students. Make sure you "see" it in both example proofs.

Your next logical question is "How in the heck am I supposed to know what statements to make or where to put them?". (See, we're mind readers.) This will come by studying examples, by experience, and in particular by knowing which of the available reasons are related to the proof at hand. Right now these will necessarily involve the algebra properties, including substitution, and the definition of betweenness of points. Try not to get discouraged as it does take time.

Our structure now begins to take on the appearance of an inverted pyramid. We still have more definitions and postulates to make, but from a relatively small beginning geometry expands rapidly and gains momentum as it grows.

EXERCISE SET 8.1

Make a single conditional statement to illustrate the chain rule in each exercise.

1. If the rain continues, then the dike will break.
   If the dike breaks, then the town will be flooded.

2. If the horseshoe does not have a nail, the shoe will be lost.
   If the shoe is lost, the horse will be lost.
   If the horse is lost, the rider will be lost.
   If the rider is lost, the battle will be lost.
   If the battle is lost, the war will be lost.

You used the chain rule in algebra every time you solved an equation. Of course, you would show the steps vertically instead of horizontally, but each step involves an implication.

Example: Given $3x + 2(x - 5) = 10$, solve for x

$$3x + 2(x-5) = 10 \rightarrow 3x + 2x - 10 = 10 \rightarrow 5x - 10 = 10 \rightarrow 5x = 20 \rightarrow x = 4$$

$$\therefore\ 3x + 2(x-5) = 10 \longrightarrow x = 4$$

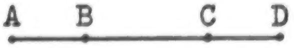
Illustrate the chain rule by solving the following equations in the form shown in the example.

3. $6x + 5 = 23$

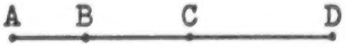
4. $3(2x + 1) - 4x = 1$

Show a complete flow proof for each of the following.

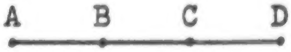
5. Given: $AC = BD$
   Prove: $AB = CD$



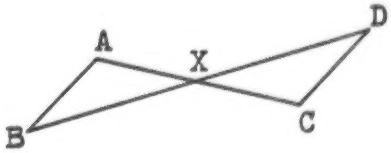
6. Given: A, B, C, and D on $\overline{AD}$ as shown
   Prove: $AB + BC + CD = AD$



7. Given: B is the midpoint of $\overline{AC}$
   C is the midpoint of $\overline{BD}$
   Prove: $AB = CD$



8. Given: $\overline{BD}$ bisects $\overline{AC}$
   $AX = AB$, $CX = CD$
   Prove: $AB = CD$



CHAPTER 9 - ANGLES AND ANGLE MEASURE
or YOU'VE GOT TO FIGURE THEM ALL!

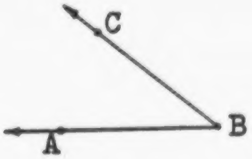
Definitions

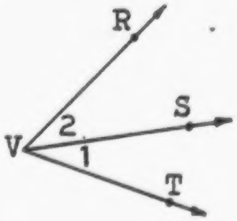
Of course, you know what an angle is, but have you ever tried to define one? Here is our formal definition for an angle.

An angle is the union of two noncollinear rays with a common endpoint.

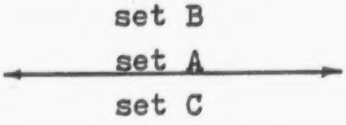
All you have to know is the meanings of all the words. The two rays are called the sides of the angle and the common endpoint is called the vertex of the angle.

There are different ways to name an angle. The first angle shown is named ∠ABC or ∠CBA. Notice, the middle letter, B, must name the vertex. When no possible confusion may result, we call it simply ∠B, meaning the angle whose vertex is at B. In the second figure, there are three angles, ∠RVS, ∠SVT, and ∠RVT. Never use a single letter to name an angle when there is more than one angle with that vertex. However, for convenience, we may name ∠SVT as ∠1 and ∠RVS as ∠2 as shown. Remember that the sides of an angle are rays and extend indefinitely.

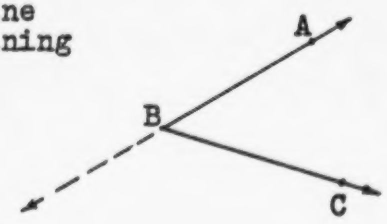




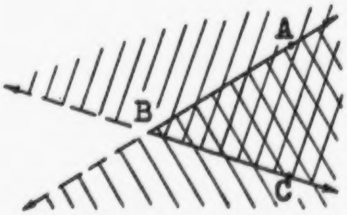
You would probably have no difficulty identifying the interior and exterior of an angle, but we do need a definition. To do this we discuss the idea of a half-plane just as we did the idea of a half-line earlier. You recall that a point separates a line into three non-intersecting sets. Likewise, a line separates a plane into three non-intersecting sets, the line itself, set A, and two sets B and C, such that if a point from B and a point from C are endpoints of a segment, the segment will intersect the line. Sets B and C are each called a half-plane. The line is called the edge of each half-plane, but it does not belong to either half-plane.



A ray by itself does not determine two half-planes, but the line containing a ray in a specific plane does determine two half-planes. Since the sides of ∠ABC are rays, $\overrightarrow{BA}$ and $\overrightarrow{BC}$, we may speak of the half-plane determined by $\overleftrightarrow{BA}$, containing C. This allows us to define the interior of an angle as follows.



The interior of ∠ABC is the intersection of the half-plane determined by $\overleftrightarrow{AB}$ containing C, with the half-plane determined by $\overleftrightarrow{BC}$ containing A.



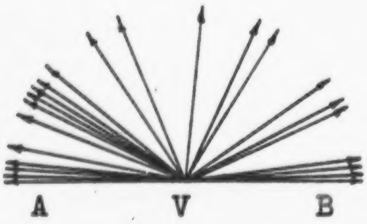
A careful reading of the definition should convince you that the interior of ∠ABC is the double shaded region in the figure, which you knew all the time. It is not a definition that you will frequently use, but it emphasizes further that a precise definition leaves no doubt as to what the thing is that is being defined.

The exterior of an angle is the set of points in the plane of the angle that do not belong to the angle itself or to its interior.

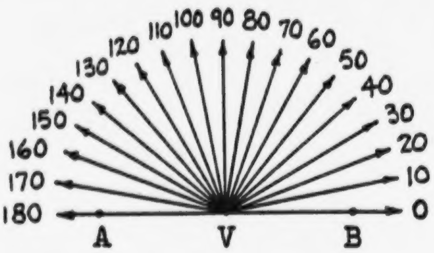
If a point lies in the interior of an angle, it is called an interior point. Notice that the set of points in the angle do not belong to either its interior or exterior.

Angle Measure

The next logical question is "How do we express the measure of an angle?", so we "invent" a reasonable method to do this just as we did with segment measure.



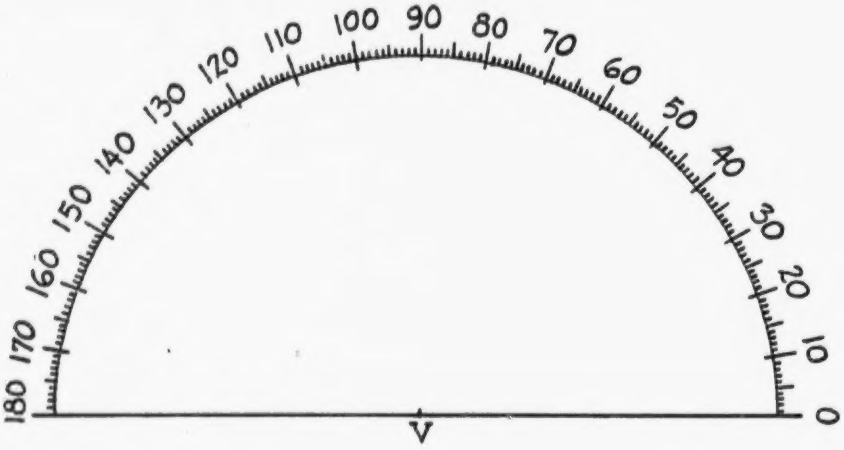
In the figure shown, let $\overleftrightarrow{AB}$ be the edge of a half-plane. Let V be any point on $\overleftrightarrow{AB}$. With V as endpoint, consider all the rays in the designated half-plane. How many are there? If you say "More than you can count!", then you are really getting the spirit of the subject. Remembering that we can't "see" the rays that we've represented, we say there are infinitely many. They are so dense that between any two, no matter how close together, there are still "more than you can count".



In the first stage all we have are rays, no numbers. Now if we decide to associate numbers with the rays, we are establishing a ray-coordinate system, just as we established a coordinate system by associating numbers with points on a line. But a major difference is that we now need a smallest number and a largest number to associate with the opposite rays, $\overrightarrow{VB}$ and $\overrightarrow{VA}$. A logical choice for the smallest number would be 0, but the largest number can be anything we wish. Different choices have been made for this largest number resulting in different units of measure for angles.

For our purposes, we'll go along with the number that is most commonly used, which is 180. Strange choice, isn't it? The choice itself appears to have originated with the Babylonians, but the reason for the choice is not known for sure. Different possibilities have been suggested.

In any case, the figure now shows $\overrightarrow{VB}$ associated with 0, $\overrightarrow{VA}$ associated with 180, and each of the rays associated with the multiples of 10 between 0 and 180. If it begins to look familiar to you, it should. We refine the "instrument" a bit more by drawing a semicircle in the half-plane, and instead of showing the entire rays, we show only the points on the semicircle that would contain the rays associated with the positive integers between 0 and 180.



This eliminates much of the clutter, and you should still be able to "see" all those rays with endpoint at V. But these rays still correspond only to the integers.

Our complete ray-coordinate system is such that there is a one to one correspondence between the rays in a half-plane with endpoint at V and the real numbers between 0 and 180. We call this our "ideal protractor".

We can consider the sides of an angle to be rays in the same half-plane, with vertex in the edge of the half-plane. The measure of the angle is the positive difference of the coordinates of those rays.

We now formally state the foregoing discussion as postulates similar to postulates 1 and 2 regarding distance and segment measure.

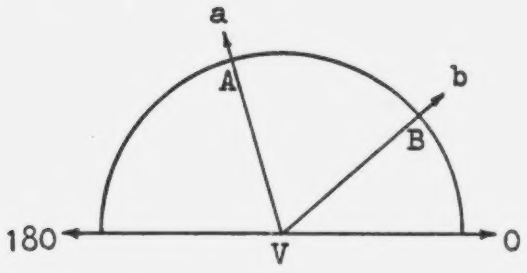
Postulate 8: To every angle there corresponds exactly one real number greater than 0 and less than 180. (Angle Measure Postulate)

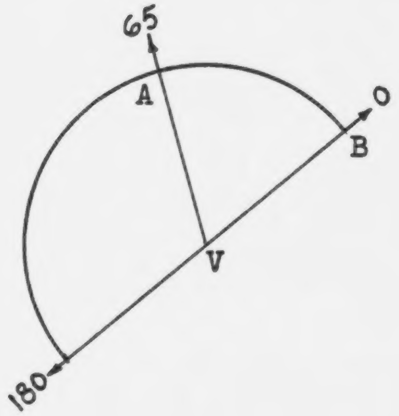
Definition: The measure of an angle is the number given by postulate 8. The measure of $\angle ABC$ is written $m\angle ABC$.

Postulate 9: In the union of a half-plane and its edge, the set of rays with a common endpoint in the edge of the half-plane can be set into correspondence with the real numbers from 0 to 180 in such a way that

1. One of the opposite rays in the edge is paired with 0 and the other is paired with 180.
2. For every ray in the set, there corresponds exactly one real number between 0 and 180.
3. For every real number between 0 and 180 there corresponds exactly one ray in the set.
4. The measure of any angle whose sides are rays in the designated set is the positive difference of the numbers corresponding to its sides. (Protractor Postulate)

As you know, the unit of measure in this system is called the degree(°). Even though it is the most commonly used unit, it is not the only one, so when we make the statement, $m\angle AVB = 65$, we are actually saying that the degree measure of $\angle AVB$ is 65. We may also say that $\angle AVB$ is a 65° angle.





The general case would be illustrated by the figure on the left in which $m\angle AVB = a - b$, $b < a$.

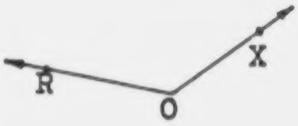
Of course, in the process of measuring an angle with your "physical protractor" you would place the protractor so the 0 ray would correspond to one side of the angle, as shown in the second figure, and read the measurement directly from the protractor at the other side of the angle.

Your protractor probably has two scales from 0 to 180 in opposite directions, which merely makes it convenient to let either side of the angle correspond to the 0 ray.

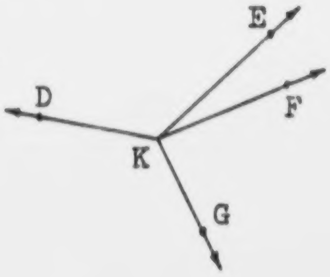
Remember that all statements made about measures and measurements of segments will also apply now to angles.

EXERCISE SET 9.1

1. An angle is the union of ________.
2. Give two different names for the angle shown.
3. Name the sides of the angle.
4. Name the vertex of the angle.

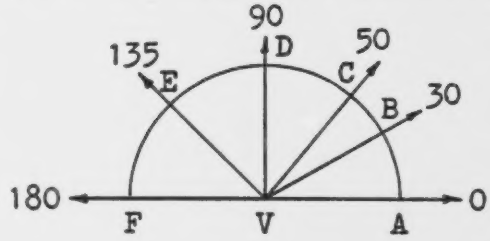


5. Name six angles in the figure shown.
6. Name a point in the interior of ∠DKF.
7. Name two points in the exterior of ∠GKF.
8. Is K in the interior of ∠EKF?
9. Is G in the exterior of ∠DKG?



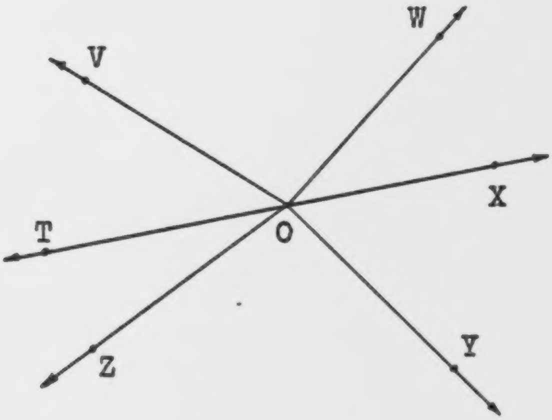
For the figure shown, give the following measures.

10. $m\angle BVC$
11. $m\angle AVE$
12. $m\angle FVC$
13. $m\angle DVB$
14. $m\angle EVD$
15. $m\angle BVE$

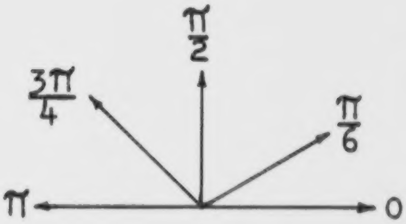


Use a protractor to measure each of the angles.

16. $m\angle WOX$
17. $m\angle YOV$
18. $m\angle TOZ$
19. $m\angle ZOX$
20. $m\angle VOT$
21. $m\angle WOY$



If two opposite rays in the edge of a half-plane are associated with the numbers 0 and π, the number associated with an angle in the half-plane is the radian measure of the angle. Thus, if the degree measure of an angle is 90, the radian measure is $\frac{\pi}{2}$.



Write the degree measure for each of the following.

22. $\frac{\pi}{6}$ 23. $\frac{3\pi}{4}$ 24. $\frac{\pi}{3}$ 25. $\frac{8\pi}{9}$

Write the radian measure for each of the following.

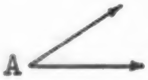
26. 45 27. 120 28. 150 29. 80

Definitions About Angles and Angle Relationships

You may already be familiar with the following words about angles, but make sure you become familiar with these definitions for them.

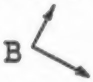
An angle is an acute angle if and only if its measure is less than 90.

$\angle A$ is an acute $\angle \longleftrightarrow m\angle A < 90$



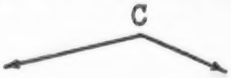
An angle is a right angle if and only if its measure is 90.

$\angle B$ is a rt. $\angle \longleftrightarrow m\angle B = 90$



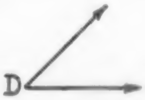
An angle is an obtuse angle if and only if its measure is greater than 90.

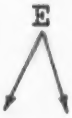
$\angle C$ is an obtuse $\angle \longleftrightarrow m\angle C > 90$



Two angles are congruent angles if and only if their measures are equal.

$\angle D \cong \angle E \longleftrightarrow m\angle D = m\angle E$

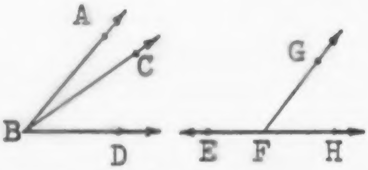




Two angles are adjacent angles if and only if they are in the same plane, have a common vertex and a common side, and have no interior points in common.

$\angle ABC$ and $\angle CBD$ are adjacent $\angle$s
$\overrightarrow{BA}$ and $\overrightarrow{BD}$ are the exterior sides

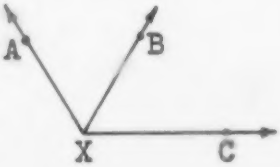
$\angle EFG$ and $\angle GFH$ are adjacent $\angle$s
$\overrightarrow{FE}$ and $\overrightarrow{FH}$ are the exterior sides



Notice that $\angle ABC$ and $\angle ABD$ are not adjacent angles because they have interior points in common.

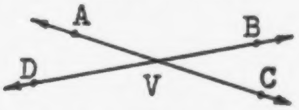
A ray is a bisector of an angle if and only if it contains an interior point of the angle and the adjacent angles formed are congruent.

$\overrightarrow{XB}$ bisects $\angle AXC \longleftrightarrow \angle AXB \cong \angle BXC$



Two angles are *vertical angles* if and only if the sides of one are opposite rays of the sides of the other.

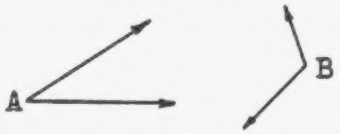
$\angle AVB$ and $\angle DVC$ are vertical $\angle$s
$\angle AVD$ and $\angle BVC$ are vertical $\angle$s



Notice that when two lines intersect, any two angles are either adjacent angles or vertical angles, but not both.

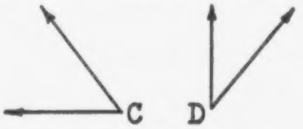
Two angles are *supplementary* if and only if the sum of their measures is 180. Each angle is the supplement of the other.

$\angle A$ suppl. $\angle B \longleftrightarrow m\angle A + m\angle B = 180$



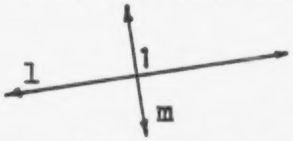
Two angles are *complementary* if and only if the sum of their measures is 90. Each angle is the complement of the other.

$\angle C$ compl. $\angle D \longleftrightarrow m\angle C + m\angle D = 90$



Two lines are *perpendicular* if and only if they form a right angle.

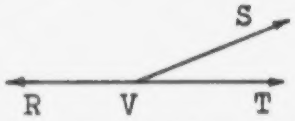
$l \perp m \longleftrightarrow \angle 1$ is a right $\angle$



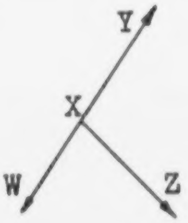
Any combination of segments and rays are perpendicular if they intersect and the lines containing them are perpendicular.

Two angles form a *linear pair* if and only if they are adjacent angles and their exterior sides are opposite rays.

$\angle RVS$ and $\angle SVT$ form a linear pair

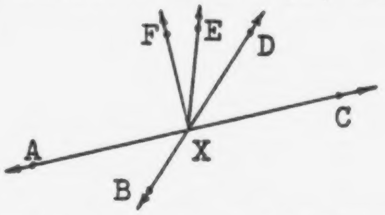


$\angle WXZ$ and $\angle ZXY$ form a linear pair



EXERCISE SET 9.2

$\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are intersecting lines. $m\angle AXF = 90$, $m\angle CXD = 45$. Name each of the following.



1. An acute angle
2. An obtuse angle
3. A right angle
4. Congruent angles
5. A pair of adjacent angles that are not a linear pair.
6. A pair of adjacent angles that are a linear pair.
7. An angle bisector
8. A pair of vertical angles
9. A second pair of vertical angles
10. A pair of supplementary angles
11. A pair of complementary angles
12. Perpendiculars
13. Name two different linear pairs for which $\overrightarrow{XE}$ is a common side.

Given the following angle measures, write the measure of the supplement of each.

14. 70   15. 150   16. n   17. 180 - n   18. m + n

Given the following angle measures, write the measure of the complement of each.

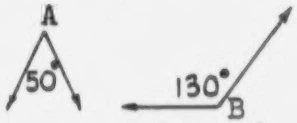
19. 25   20. 85   21. n   22. 90 - n   23. r + s

24. The measure of an angle is 40 more than the measure of its supplement. Find the measure of the angle.

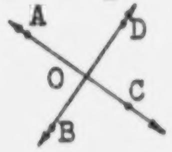
25. The measure of an angle is twice the measure of its complement. Find the measure of the angle.

26. The measure of the supplement of an angle is three times the measure of its complement. Find the measure of the angle.

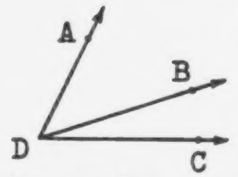
27. Make the three statements in the Law of Detachment to conclude the angles shown are supplementary.



28. Given that $\angle BOC$ is a right angle, make the three statements in the Law of Detachment to conclude that $\overleftrightarrow{AC} \perp \overleftrightarrow{BD}$.



Theorems About Angles



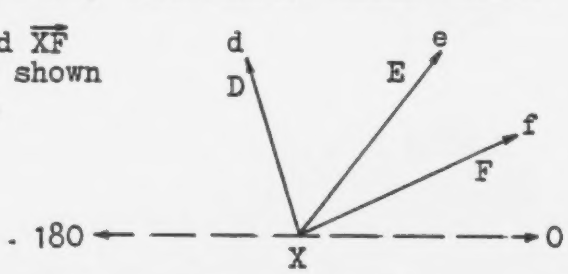
Definition: $\overrightarrow{DB}$ lies between $\overrightarrow{DA}$ and $\overrightarrow{DC}$ if and only if all three rays lie in the same half-plane and B is an interior point of $\angle ADC$.

The following theorem will allow us to add angle measures just as the definition of betweenness of points allows us to add segment lengths. We will refer to it by the name Angle Addition Theorem.

Theorem 9-1: If $\overrightarrow{XE}$ is between $\overrightarrow{XD}$ and $\overrightarrow{XF}$, and the coordinates of $\overrightarrow{XD}$, $\overrightarrow{XE}$, and $\overrightarrow{XF}$ are d, e, and f respectively with $f < e$ and $e < d$, then $m\angle DXE + m\angle EXF = m\angle DXF$.

Given: $\overrightarrow{XE}$ is between $\overrightarrow{XD}$ and $\overrightarrow{XF}$ with coordinates as shown

Prove: $m\angle DXE + m\angle EXF = m\angle DXF$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ m\angle DXE = d - e \\ \textcircled{2}\ m\angle EXF = e - f \end{array}\right\} \xrightarrow{\textcircled{3}} m\angle DXE + m\angle EXF = (d - e) + (e - f) \xrightarrow{\textcircled{4}}$$

$$\left.\begin{array}{r} \xrightarrow{\textcircled{4}} m\angle DXE + m\angle EXF = d - f \\ \textcircled{5}\ m\angle DXF = d - f \end{array}\right\} \xrightarrow{\textcircled{6}} m\angle DXE + m\angle EXF = m\angle DXF$$

$\therefore$ $\overrightarrow{XE}$ is between $\overrightarrow{XD}$ and $\overrightarrow{XF} \longrightarrow m\angle DXE + m\angle EXF = m\angle DXF$

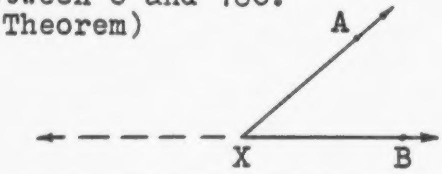
① Protractor postulate
② Same as ①
③ Add. prop. of =
④ By algebra $(d - e) + (e - f) = d - e + e - f = d - f$
⑤ Same as ①
⑥ Transitive prop. of =

We don't require the statement, $\overrightarrow{XE}$ is between $\overrightarrow{XD}$ and $\overrightarrow{XF}$, to be stated in the argument, but it is a necessary condition.

We will state two existence and uniqueness theorems related to angles that are similar in form to theorems 7-1 and 7-2 about points and segments. The proofs would be similar in structure to the informal proofs of the earlier theorems. We will not present tnem here.

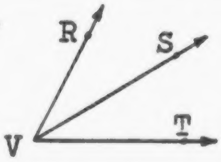
Theorem 9-2: Through the endpoint of a ray lying in the edge of a half-plane, there is exactly one other ray in the half-plane such that the measure of the angle formed by the two rays is a given number between 0 and 180. (Angle Construction Theorem)

If n is the given number, then $\overrightarrow{XA}$ is a unique ray such that m∠BXA = n.



Theorem 9-3: An angle has exactly one bisector.

If $\overrightarrow{VS}$ bisects ∠RVT, it is the only bisector.

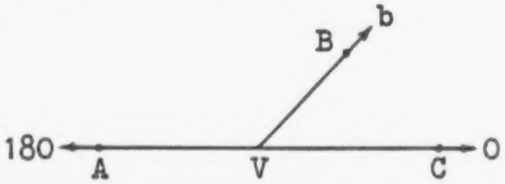


The following theorem is also quite obvious, but it is one that you will use very frequently since this condition is present in many figures.

Theorem 9-4: If two angles are a linear pair, they are supplementary.

Given: ∠AVB and ∠BVC are a linear pair

Prove: ∠AVB suppl. ∠BVC



Argument

Since $\overrightarrow{VC}$ and $\overrightarrow{VA}$ are opposite rays, let $\overrightarrow{VC}$ correspond to 0, $\overrightarrow{VA}$ correspond to 180, and $\overrightarrow{VB}$ correspond to any number b, between 0 and 180.

$$\left.\begin{array}{l} \textcircled{1}\ m\angle AVB = 180 - b \\ \textcircled{2}\ m\angle BVC = b - 0 \end{array}\right\} \xrightarrow{\textcircled{3}} m\angle AVB + m\angle BVC = (180 - b) + (b - 0) \xrightarrow{\textcircled{4}}$$

$$\xrightarrow{\textcircled{4}} m\angle AVB + m\angle BVC = 180 \xrightarrow{\textcircled{5}} \angle AVB \text{ suppl. } \angle BVC$$

$\therefore$ ∠AVB and ∠BVC are a linear pair ⟶ ∠AVB suppl. ∠BVC

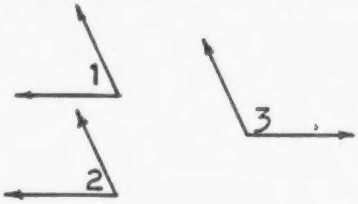
① Protractor postulate
② Same as ①
③ Add. prop. of =
④ By algebra $(180 - b) + (b - 0) = 180 - b + b - 0 = 180$
⑤ If the sum of the measures of 2 ∠s is 180, then they are supplementary.

Theorems About Angle Relations

Theorem 9-5: If two angles are supplementary to the same angle, then they are congruent to each other.

Given: $\angle 1$ suppl. $\angle 3$
$\angle 2$ suppl. $\angle 3$

Prove: $\angle 1 \cong \angle 2$



Argument

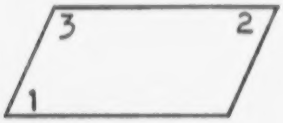
$$\left.\begin{array}{l} \textcircled{1}\ \angle 1 \text{ suppl. } \angle 3 \xrightarrow{\textcircled{2}} m\angle 1 + m\angle 3 = 180 \\ \textcircled{3}\ \angle 2 \text{ suppl. } \angle 3 \xrightarrow{\textcircled{4}} m\angle 2 + m\angle 3 = 180 \end{array}\right\} \xrightarrow{\textcircled{5}} m\angle 1 + m\angle 3 = m\angle 2 + m\angle 3 \xrightarrow{\textcircled{6}}$$

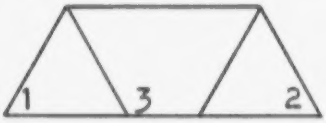
$$\xrightarrow{\textcircled{6}} m\angle 1 = m\angle 2 \xrightarrow{\textcircled{7}} \angle 1 \cong \angle 2$$

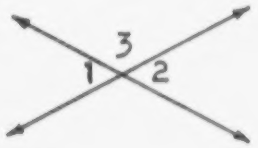
$$\therefore \left.\begin{array}{l} \angle 1 \text{ suppl. } \angle 3 \\ \angle 2 \text{ suppl. } \angle 3 \end{array}\right\} \longrightarrow \angle 1 \cong \angle 2$$

① Given
② If 2 ∠s are suppl., the sum of their measures is 180.
③ Given
④ Same as ②
⑤ Transitive prop. of =
⑥ Subtraction prop. of =
⑦ If two angles have equal measures, they are ≅.

A comment about the generalization aspect of proof is in order. To present an argument for the proof, we have drawn a specific figure with specific names for the angles. However, when we state the implication following statement ⑦, we are stating it for the infinite number of possible ways in which the statements to the left of the arrow can occur. Very seldom will the figure in which you apply a theorem as a reason look like the figure in which you proved it to begin with. For example, if the two statements to the left of the arrow are established in some way in the following figures, then in each case we may conclude $\angle 1 \cong \angle 2$.







This is how your superior powers of observation will play a role in recognizing when a theorem that you've already proved can be applied to a completely new problem. So every time a figure is drawn to help prove a theorem, it should be thought of as merely a "representative" of the entire infinite set of figures that satisfy the conditions of the hypothesis.

The next three theorems are similar in form and structure.

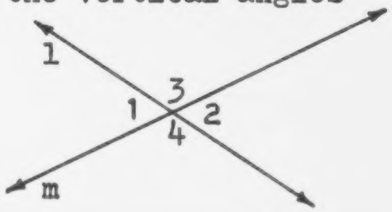
Theorem 9-6: If two angles are supplementary to congruent angles, then they are congruent to each other.

Theorem 9-7: If two angles are complementary to the same angle, then they are congruent to each other.

Theorem 9-8: If two angles are complementary to congruent angles, then they are congruent to each other.

For the next theorem, we're hoping you caught the hint in the discussion following theorem 9-5.

Theorem 9-9: If two lines intersect, the vertical angles are congruent.



Given: l and m intersect as shown

Prove: $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$

Argument

$$\left.\begin{array}{l} \text{①} \ \angle 1 \text{ suppl. } \angle 3 \\ \text{②} \ \angle 2 \text{ suppl. } \angle 3 \end{array}\right\} \xrightarrow{\text{③}} \angle 1 \cong \angle 2$$

Similarily, $\angle 3 \cong \angle 4$

$\therefore$ l and m intersect $\longrightarrow \angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$

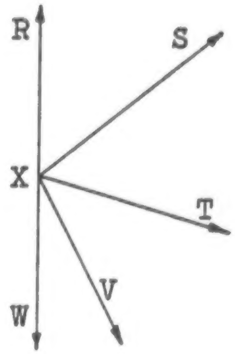
① If 2 ∠s are a linear pair, they are suppl.
② Same as ①
③ If 2 ∠s are suppl. to the same ∠, they are ≅.

When using this theorem as a reason later, you may state simply "vertical angles are congruent". It is not necessary to state that two lines intersect in the argument of a proof if it is shown in the figure.

EXERCISE SET 9.3

Given R, X, and W are collinear. $\overrightarrow{XS}$ bisects $\angle RXT$. Write the theorem that would justify statements 1-4.

1. Consider the half plane with edge $\overleftrightarrow{RW}$ and containing T. With $\overrightarrow{XR}$ as one side of an angle, there is exactly one other ray in the half-plane such that the measure of the angle formed by the two rays is equal to $m\angle WXV$.
2. $\angle RXT$ and $\angle TXW$ are supplementary
3. $m\angle TXV + m\angle VXW = m\angle TXW$
4. $\overrightarrow{XS}$ is the only bisector of $\angle RXT$.

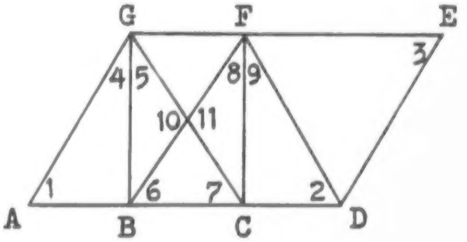


Ex. 1-8

5. If $m\angle RXS = 50$ and $m\angle SXV = 110$, then $m\angle RXV = ?$
6. If $m\angle WXT = 70$ and $m\angle WXS = 135$, then $m\angle TXS = ?$
7. If $m\angle VXT = 2x - 3$ and $m\angle TXS = x + 5$, then $m\angle VXS = ?$
8. Give the reasons for both statements ① and ② in the following implication.

   ① $\angle WXS$ suppl. $\angle SXR \xrightarrow{②} m\angle WXS + m\angle SXR = 180$

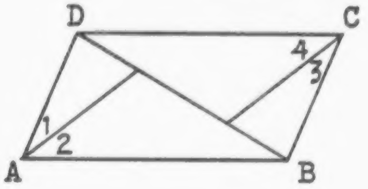
Using the figure shown, write the theorem that would justify each of the following.



9. $\left.\begin{array}{r}\angle 7 \text{ compl. } \angle 5\\ \angle 6 \text{ compl. } \angle 8\\ \angle 5 \cong \angle 8\end{array}\right\} \longrightarrow \angle 7 \cong \angle 6$
10. $\left.\begin{array}{r}\angle 1 \text{ suppl. } \angle AGE\\ \angle 2 \text{ suppl. } \angle AGE\end{array}\right\} \longrightarrow \angle 1 \cong \angle 2$
11. $\angle 10 \cong \angle 11$
12. $\left.\begin{array}{r}\angle 4 \text{ compl. } \angle 1\\ \angle 9 \text{ compl. } \angle 1\end{array}\right\} \longrightarrow \angle 4 \cong \angle 9$
13. $\left.\begin{array}{r}\angle AGE \text{ suppl. } \angle 3\\ \angle ADE \text{ suppl. } \angle 1\\ \angle 3 \cong \angle 1\end{array}\right\} \longrightarrow \angle AGE \cong \angle ADE$

14. Write the reason for each statement in the following proof.

Given: ∠1 ≅ ∠3, ∠2 ≅ ∠4
Prove: ∠DAB ≅ ∠BCD



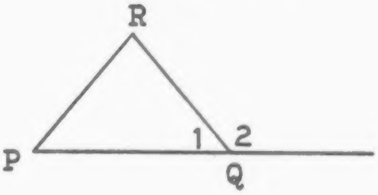
Argument

$$\left.\begin{array}{l} \text{①}\ \angle 1 \cong \angle 3 \xrightarrow{\text{②}} m\angle 1 = m\angle 3 \\ \text{③}\ \angle 2 \cong \angle 4 \xrightarrow{\text{④}} m\angle 2 = m\angle 4 \end{array}\right\} \xrightarrow{\text{⑤}} \left.\begin{array}{l} m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4 \\ \text{⑥}\ m\angle 1 + m\angle 2 = m\angle DAB \\ \text{⑦}\ m\angle 3 + m\angle 4 = m\angle BCD \end{array}\right\} \xrightarrow{\text{⑧}}$$

$$\xrightarrow{\text{⑧}} m\angle DAB = m\angle BCD \xrightarrow{\text{⑨}} \angle DAB \cong \angle BCD$$

15. All the statements in the argument of the following proof are written below. Write the statements in proper flow proof form as indicated. (Hint: Try working backwards)

Given: m∠P = 50, m∠2 = 130
Prove: ∠1 ≅ ∠P



Argument

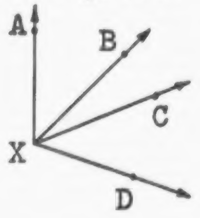
$$\text{①}\ ?\ \left.\begin{array}{l} \xrightarrow{\text{②}}\ ? \\ \quad\ \text{③}\ \ ? \end{array}\right\} \xrightarrow{\text{④}}\ ?\ \left.\begin{array}{l} \xrightarrow{\text{⑤}}\ ? \\ \quad\ \text{⑥}\ \ ? \end{array}\right\} \xrightarrow{\text{⑦}}\ ? \xrightarrow{\text{⑧}}\ ?$$

a) m∠1 + 130 = 180
b) m∠1 = m∠P
c) m∠1 + m∠2 = 180
d) m∠2 = 130
e) ∠1 suppl. ∠2
f) m∠P = 50
g) ∠1 ≅ ∠P
h) m∠1 = 50

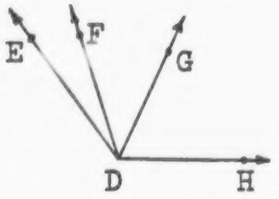
16. Give the reason for each statement of exercise 15.

In all proof exercises, always copy the figure, the "given", and the "prove", and then complete the proof.

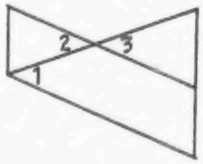
17. Given: $m\angle AXB = m\angle CXD$
Prove: $m\angle AXC = m\angle BXD$



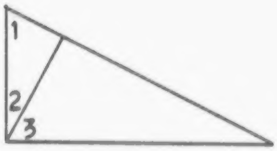
18. Given: Figure as shown
Prove: $m\angle EDF + m\angle FDG + m\angle GDH = m\angle EDH$



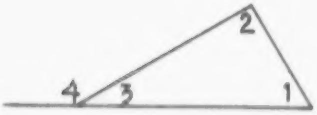
19. Given: $m\angle 1 = m\angle 3$
Prove: $\angle 1 \cong \angle 2$



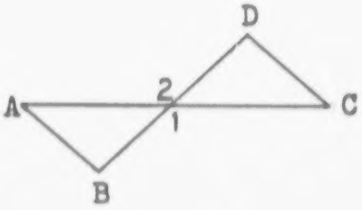
20. Given: $m\angle 1 + m\angle 2 = 90$
$\angle 2$ compl. $\angle 3$
Prove: $\angle 1 \cong \angle 3$



21. Given: $m\angle 1 = 60$, $m\angle 2 = 90$, $m\angle 3 = 30$
Prove: $m\angle 4 = m\angle 1 + m\angle 2$



22. Given: $\angle A$ suppl. $\angle 1$
$\angle C$ suppl. $\angle 2$
Prove: $\angle A \cong \angle C$



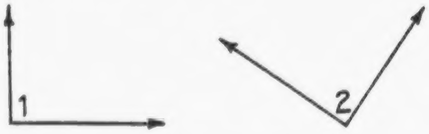
Theorems About Perpendicular Lines and Right Angles

There is a strong temptation to assume lines are perpendicular from the appearance of a figure, but this is not permissible. Perpendicular lines and right angles must either be given or proved. There are a number of theorems about these which are quite helpful.

Theorem 9-10: All right angles are congruent.

Given: ∠1 and ∠2 are rt. ∠s

Prove: ∠1 ≅ ∠2



Argument

$$\left.\begin{array}{l} \text{①}\ \angle 1 \text{ is a rt.}\angle \xrightarrow{\text{②}} m\angle 1 = 90 \\ \text{③}\ \angle 2 \text{ is a rt.}\angle \xrightarrow{\text{④}} m\angle 2 = 90 \end{array}\right\} \xrightarrow{\text{⑤}} m\angle 1 = m\angle 2 \xrightarrow{\text{⑥}} \angle 1 \cong \angle 2$$

$$\therefore \left.\begin{array}{l} \angle 1 \text{ is a rt.}\angle \\ \angle 2 \text{ is a rt.}\angle \end{array}\right\} \longrightarrow \angle 1 \cong \angle 2$$

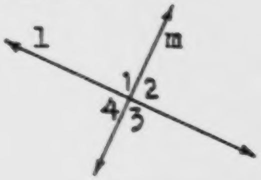
① Given
② If an ∠ is a rt.∠, its measure is 90.
③ Given
④ Same as ②
⑤ Transitive prop. of =
⑥ If 2 ∠s have = measures, they are ≅.

Reasons ②, ④, and ⑥ are, of course, definitions. We have written them here as conditional statements to continue to emphasize the Law of Detachment. Since a biconditional definition can be stated two ways, you must recognize which of these two should be used as a reason in a particular case. It depends on the order of the statements so that the Law of Detachment will apply. When you are absolutely sure that you understand this relationship, then you may write these reasons as follows.

② Def. of a rt. ∠

⑥ Def. of ≅ ∠s

Theorem 9-11: If two lines are perpendicular, then all four angles are right angles.



Given: $l \perp m$

Prove: ∠1, ∠2, ∠3, and ∠4 are rt. ∠s

Argument

$$\left.\begin{array}{l} \text{①}\ l \perp m \xrightarrow{\text{②}} \angle 1 \text{ is a rt.} \angle \xrightarrow{\text{③}} m\angle 1 = 90 \\ \quad \text{④}\ \angle 1 \text{ suppl. } \angle 2 \xrightarrow{\text{⑤}} m\angle 1 + m\angle 2 = 180 \end{array}\right\} \xrightarrow{\text{⑥}} 90 + m\angle 2 = 180 \xrightarrow{\text{⑦}}$$

$$\xrightarrow{\text{⑦}} m\angle 2 = 90 \xrightarrow{\text{⑧}} \angle 2 \text{ is a rt.} \angle$$

$$\left.\begin{array}{r} \text{⑨}\ \angle 1 \cong \angle 3 \xrightarrow{\text{⑩}} m\angle 1 = m\angle 3 \\ \text{③}\ m\angle 1 = 90 \end{array}\right\} \xrightarrow{\text{⑪}} m\angle 3 = 90 \xrightarrow{\text{⑫}} \angle 3 \text{ is a rt.}$$

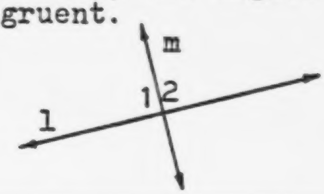
Similarly, ∠4 is a rt.∠ by either of the above arguments.

$\therefore\ l \perp m \longrightarrow \angle 1, \angle 2, \angle 3,$ and $\angle 4$ are rt. ∠s

① Given
② If two lines are ⊥, they form a rt. ∠.
③ If an ∠ is a rt. ∠, its measure is 90.
④ If 2 ∠s are a linear pair, they are suppl.
⑤ If 2 ∠s are suppl., the sum of their measures is 180.
⑥ Substitution
⑦ Subtraction prop. of =
⑧ If the measure of an ∠ is 90, it is a rt. ∠.
⑨ Vertical ∠s are ≅
⑩ If 2 ∠s are ≅, their measures are =
⑪ Transitive prop. of =
⑫ Same as ⑧

Notice that when a statement is repeated later in a proof, such as statement ③, we merely rewrite it with the same number since it has already been established.

Theorem 9-12: If two lines are perpendicular, the angles in a linear pair are congruent.



Given: $l \perp m$

Prove: $\angle 1 \cong \angle 2$

Argument

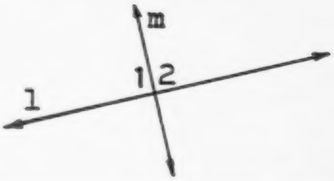
$$\begin{tikzcd} & \angle 1 \text{ is a rt. } \angle \arrow[dr, "\text{④}"] & \\ \text{①}\ l \perp m \arrow[ur, "\text{②}"] \arrow[dr, "\text{③}"'] & & \angle 1 \cong \angle 2 \\ & \angle 2 \text{ is a rt. } \angle \arrow[ur] & \end{tikzcd}$$

$\therefore l \perp m \longrightarrow \angle 1 \cong \angle 2$

① Given
② If 2 lines are ⊥, all 4 ∠s are rt. ∠s.
③ Same as ②
④ All rt. ∠s are ≅

Next is the converse of Theorem 9-12. Remember that converses are not equivalent, and even though they may both be true, they must be proved independently of one another.

Theorem 9-13: If two lines intersect so the angles in a linear pair are congruent, the lines are perpendicular.



Given: $\angle 1 \cong \angle 2$

Prove: $l \perp m$

Argument

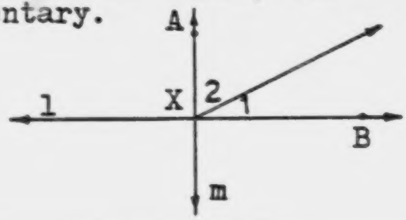
$$\begin{tikzcd} \text{①}\ \angle 1 \text{ suppl. } \angle 2 \arrow[r, "\text{②}"] & m\angle 1 + m\angle 2 = 180 \arrow[dr] & \\ \text{③}\ \angle 1 \cong \angle 2 \arrow[r, "\text{④}"] & m\angle 1 = m\angle 2 \arrow[r, "\text{⑤}"] & m\angle 1 + m\angle 1 = 180 \xrightarrow{\text{⑥}} \end{tikzcd}$$

$$\xrightarrow{\text{⑥}} 2m\angle 1 = 180 \xrightarrow{\text{⑦}} m\angle 1 = 90 \xrightarrow{\text{⑧}} \angle 1 \text{ is a rt. } \angle \xrightarrow{\text{⑨}} l \perp m$$

$\therefore \angle 1 \cong \angle 2 \longrightarrow l \perp m$

① If 2 ∠s are a linear pair, they are suppl.
② If 2 ∠s are suppl. the sum of their measures is 180.
③ Given
④ If 2 ∠s are ≅, their measures are =.
⑤ Substitution
⑥ Combining like terms
⑦ Division prop. of =
⑧ If the measure of an angle is 90, it is a rt. ∠.
⑨ If 2 lines intersect to form a rt. ∠, they are ⊥.

Theorem 9-14: If the exterior sides of two adjacent acute angles are in perpendicular lines, the angles are complementary.



Given: $l \perp m$

Prove: ∠1 compl. ∠2

Argument

$$① \ l \perp m \xrightarrow{\text{②}} \angle AXB \text{ is a rt. } \angle \xrightarrow{\text{③}} \left.\begin{array}{r} m\angle AXB = 90 \\ \text{④}\ m\angle 1 + m\angle 2 = m\angle AXB \end{array}\right\} \xrightarrow{\text{⑤}}$$

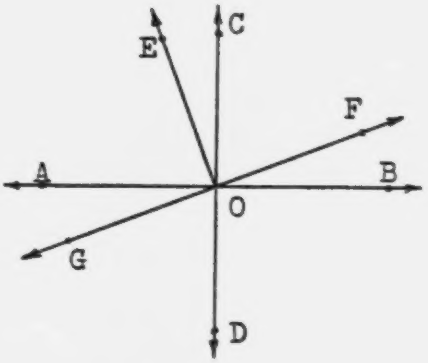
$$\xrightarrow{\text{⑤}} m\angle 1 + m\angle 2 = 90 \xrightarrow{\text{⑥}} \angle 1 \text{ compl. } \angle 2$$

$$\therefore l \perp m \longrightarrow \angle 1 \text{ compl. } \angle 2$$

① Given
② If 2 lines are ⊥, they form a rt. ∠.
③ If an ∠ is a rt. ∠, its measure is 90.
④ ∠ addition thm.
⑤ Transitive prop. of =
⑥ If the sum of the measures of 2 ∠s is 90, they are compl.

EXERCISE SET 9.4

Using the figure shown, complete each implication.



1. $\overleftrightarrow{AB} \perp \overleftrightarrow{CD} \begin{array}{l} \nearrow \angle ? \text{ compl. } \angle ? \\ \searrow \angle ? \text{ compl. } \angle ? \end{array}$

2. $\angle COA \cong \angle AOD \longrightarrow ? \perp ?$

3. $\left.\begin{array}{l} \angle EOF \text{ is a rt. } \angle \\ \angle COB \text{ is a rt. } \angle \end{array}\right\} \longrightarrow ?$

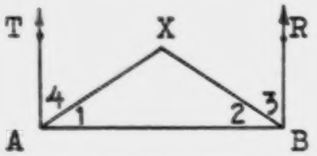
4. $\overrightarrow{OE} \perp \overrightarrow{OF} \longrightarrow \angle ? \text{ compl. } \angle ?$

5. $\overleftrightarrow{GF} \perp \overrightarrow{OE} \longrightarrow \angle ? \cong \angle ?$

Two sets of statements are made in the argument of the following proof. They are both acceptable but the second set is obviously more direct. This illustrates the advantage of being familiar with and making use of all theorems which may relate to a specific example.

6. Give the reasons for all statements in both sets.

Given: $\overrightarrow{BR} \perp \overline{BA}$, $\overrightarrow{AT} \perp \overline{AB}$, $\angle 1 \cong \angle 2$
Prove: $\angle 3 \cong \angle 4$



Argument 1

$$\left.\begin{array}{l} \text{①}\ \overrightarrow{BR} \perp \overline{BA} \xrightarrow{\text{②}} \angle ABR \text{ is a rt. } \angle \\ \text{③}\ \overrightarrow{AT} \perp \overline{AB} \xrightarrow{\text{④}} \angle BAT \text{ is a rt. } \angle \end{array}\right\} \xrightarrow{\text{⑤}} \angle ABR \cong \angle BAT \xrightarrow{\text{⑥}}$$

$$\left.\begin{array}{l} \xrightarrow{\text{⑥}} m\angle ABR = m\angle BAT \\ \text{⑦}\ m\angle ABR = m\angle 2 + m\angle 3 \\ \text{⑧}\ m\angle BAT = m\angle 1 + m\angle 4 \end{array}\right\} \left.\begin{array}{l} \xrightarrow{\text{⑨}} m\angle 2 + m\angle 3 = m\angle 1 + m\angle 4 \\ \text{⑩}\ \angle 1 \cong \angle 2 \xrightarrow{\text{⑪}} m\angle 1 = m\angle 2 \end{array}\right\} \xrightarrow{\text{⑫}}$$

$$\xrightarrow{\text{⑫}} m\angle 1 + m\angle 3 = m\angle 1 + m\angle 4 \xrightarrow{\text{⑬}} m\angle 3 = m\angle 4 \xrightarrow{\text{⑭}} \angle 3 \cong \angle 4$$

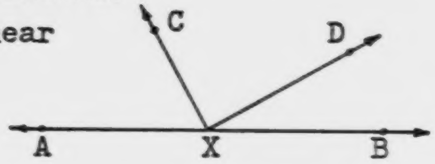
Argument 2

$$\left.\begin{array}{l} \text{①}\ \overrightarrow{BR} \perp \overline{BA} \xrightarrow{\text{②}} \angle 2 \text{ compl. } \angle 3 \\ \text{③}\ \overrightarrow{AT} \perp \overline{AB} \xrightarrow{\text{④}} \angle 1 \text{ compl. } \angle 4 \\ \qquad\qquad\qquad \text{⑤}\ \angle 1 \cong \angle 2 \end{array}\right\} \xrightarrow{\text{⑥}} \angle 3 \cong \angle 4$$

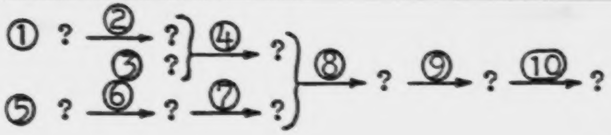
7. All the statements in the argument of the following proof are written below. Write the statements in proper flow proof form as indicated.

Given: A, X, and B are collinear
$\overrightarrow{XC} \perp \overrightarrow{XD}$

Prove: ∠AXC compl. ∠BXD

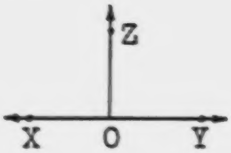


Argument

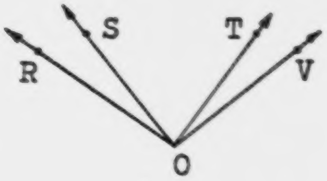


a) $\overline{XC} \perp \overline{XD}$	f) m∠AXC + m∠CXD + m∠BXD = 180
b) m∠AXC + m∠BXD = 90	g) ∠AXC compl. ∠BXD
c) m∠CXB = m∠CXD + m∠BXD	h) m∠AXC + 90 + m∠BXD = 180
d) m∠CXD = 90	i) ∠CXD is a rt. ∠
e) ∠AXC suppl. ∠CXB	j) m∠AXC + m∠CXB = 180

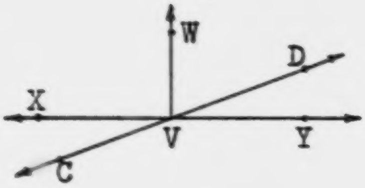
8. Given: ∠YOZ is a rt. ∠
Prove: ∠YOZ ≅ ∠XOZ



9. Given: $\overrightarrow{OR} \perp \overrightarrow{OT}$, $\overrightarrow{OS} \perp \overrightarrow{OV}$
Prove: ∠ROS ≅ ∠TOV

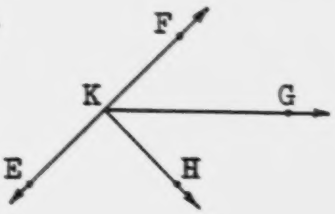


10. Given: $\overleftrightarrow{XY} \perp \overrightarrow{VW}$, m∠XVC = 20
Prove: m∠DVW = 70



11. Given: E, K, and F are collinear
$\overrightarrow{KG}$ bisects ∠FKH
m∠FKG = 45

Prove: $\overleftrightarrow{EF} \perp \overrightarrow{KH}$



Reflexive, Symmetric, and Transitive Properties

The three properties of equality, reflexive, symmetric, and transitive, are very significant in mathematics. They will hold for other relations besides equality, and we will present them here for congruence. For equality, they are postulates. For congruence, they will be theorems in which we use the equality postulates in the proof. We'll show them for congruence of segments, and let you show them for congruence of angles.

Theorem 9-15: Congruence of segments is reflexive.

Given: $\overline{AB}$

Prove: $\overline{AB} \cong \overline{AB}$

Argument

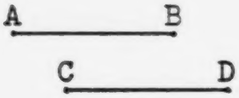
$$\textcircled{1}\ AB = AB \xrightarrow{\textcircled{2}} \overline{AB} \cong \overline{AB}$$

① Reflexive prop. of =
② Def. of ≅ segments

Theorem 9-16: Congruence of segments is symmetric.

Given: $\overline{AB} \cong \overline{CD}$

Prove: $\overline{CD} \cong \overline{AB}$



Argument

$$\textcircled{1}\ \overline{AB} \cong \overline{CD} \xrightarrow{\textcircled{2}} AB = CD \xrightarrow{\textcircled{3}} CD = AB \xrightarrow{\textcircled{4}} \overline{CD} \cong \overline{AB}$$

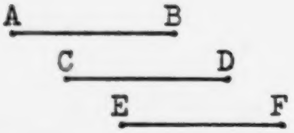
$$\therefore\ \overline{AB} \cong \overline{CD} \longrightarrow \overline{CD} \cong \overline{AB}$$

You supply the reasons.

Theorem 9-17: Congruence of segments is transitive.

Given: $\overline{AB} \cong \overline{CD}$, $\overline{CD} \cong \overline{EF}$

Prove: $\overline{AB} \cong \overline{EF}$



Argument

$$\left.\begin{array}{l}\textcircled{1}\ \overline{AB} \cong \overline{CD} \xrightarrow{\textcircled{2}} AB = CD \\ \textcircled{3}\ \overline{CD} \cong \overline{EF} \xrightarrow{\textcircled{4}} CD = EF\end{array}\right\} \xrightarrow{\textcircled{5}} AB = EF \xrightarrow{\textcircled{6}} \overline{AB} \cong \overline{EF}$$

$$\therefore\ \left.\begin{array}{l}\overline{AB} \cong \overline{CD} \\ \overline{CD} \cong \overline{EF}\end{array}\right\} \longrightarrow \overline{AB} \cong \overline{EF}$$

You supply the reasons.

Theorem 9-18: Congruence of angles is reflexive.

Theorem 9-19: Congruence of angles is symmetric.

Theorem 9-20: Congruence of angles is transitive.

EXERCISE SET 9.5

1. Prove Theorem 9-18.
2. Prove Theorem 9-19.
3. Prove Theorem 9-20.

CHAPTER 10 - INDIRECT PROOF

or JUST SUPPOSE

Indirect Reasoning

You have surely used this "backdoor" type of reasoning at some time. Carl brings his friend, Mark, home with him, and as they enter the house, Carl sees sister Emily in a new dress.

Carl: A-hah! Emily has a date tonight.
Mark: How do you know?
Carl: Well, if she didn't have a date, then she wouldn't be wearing her new dress.

We call this *indirect reasoning*. Notice that Carl's last statement is *the contrapositive* of the statement "If Emily is wearing her new dress, then she has a date.". This is a very simple example of indirect reasoning. Remember that a statement and its contrapositive are equivalent. It is sometimes convenient to establish the truth of a statement by showing the truth of its contrapositive.

The general approach to indirect proof involves assuming that the statement you want to prove is false and seeing what that leads to. In Carl's case, assuming that she didn't have a date led him to conclude that she wouldn't be wearing her new dress. But this contradicted the fact that she was wearing her new dress. So he must now conclude that his assumption was false, in other words that she does have a date.

We will discuss an important principle related to indirect proof, the *Law of Contradiction*.

The two statements, p and ~p, cannot both be true or both be false at the same time.

In symbols: $(p \wedge \sim p) \longrightarrow$ p is true and ~p is false, or
p is false and ~p is true

We say that the statements, p and ~p, are contradictory. We are particularly concerned with the fact that one of them *must* be false.

We will always begin an indirect proof by saying, "Suppose the statement we wish to prove is false.". This is the same as saying, "Suppose the negation of the statement we wish to prove is true.". We reason from this statement deductively until we arrive at a statement which contradicts a statement known to be true. Then we must conclude that the supposition which led to the contradiction is false. That is, "It is false that the statement we wish to prove is false.". This doubletalk means, of course, that it is true.

We can show this symbolically as indicated below.

Prove: p

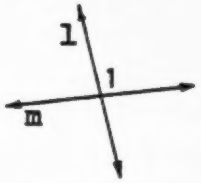
Suppose ~p is true

$$\left.\begin{array}{l} \sim p \longrightarrow q \longrightarrow r \longrightarrow s \longrightarrow t \\ \qquad\qquad\qquad \text{but } \sim t \text{ is true} \end{array}\right\} \longrightarrow \sim p \text{ is false}$$

$$\therefore p \text{ is true}$$

The word "suppose" is a key word. We're not actually saying ~p is true. We just want to see what happens if we assume that it is. Even though t must be false in this case, it is still a valid conclusion. So the breakdown that led to the contradiction had to occur at the beginning with the supposition. Reread this, if necessary, after you've studied some examples.

Certain types of proofs can be approached more reasonably by either the direct application of the contrapositive or by using indirect proof. First, an example using the contrapositive.



Prove: If $\angle 1$ is not a rt. $\angle$, then $l \not\perp m$

($\not\perp$ means "is not perpendicular to")

The contrapositive of this statement is

If $l \perp m$, then $\angle 1$ is a rt. $\angle$.

We know the latter statement is true by the definition of perpendicular lines. Therefore, the first statement is also true since the two statements are equivalent.

The more general method of indirect proof is rather difficult to learn, and takes time to absorb. We'll look at an example from algebra first.

Prove: If $a > 0$ and $b > 0$, then $(a+b)^2 \neq a^2 + b^2$

Suppose $(a+b)^2 = a^2 + b^2$

$$\textcircled{1}\ (a+b)^2 = a^2 + b^2 \xrightarrow{\textcircled{2}} a^2 + 2ab + b^2 = a^2 + b^2 \xrightarrow{\textcircled{3}}$$

$$\left.\begin{array}{l} \xrightarrow{\textcircled{3}} 2ab = 0 \xrightarrow{\textcircled{4}} a = 0 \text{ or } b = 0 \\ \textcircled{5}\ \text{but } a > 0 \text{ and } b > 0 \text{ is true} \end{array}\right\} \xrightarrow{\textcircled{6}} \begin{array}{l} (a+b)^2 = a^2 + b^2 \\ \text{is false} \end{array}$$

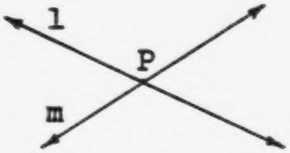
$$\therefore (a+b)^2 \neq a^2 + b^2$$

The reasons for ②, ③, and ④ are algebra, which we won't write. Reason ① is supposition, ⑤ is given, and ⑥ is **Law of Contradiction.** Notice that all reasoning following the supposition is direct until we reach the contradiction of a known fact, in this case, the hypothesis.

One of our first theorems, Thm. 7-3, was not proved earlier because its proof is indirect. It is a somewhat difficult proof of a fairly simple theorem. We'll consider it here as an example of indirect proof in geometry.

Theorem 7-3: If two different lines intersect, they intersect in exactly one point.

Given: l and m intersect at P
Prove: P is the only point of intersection



Argument

Suppose l and m intersect in a second point, Q.

$$\left.\begin{array}{l}\text{① l contains P and Q} \xrightarrow{②} \text{l is only line through P and Q}\\ \text{③ m contains P and Q} \xrightarrow{④} \text{m is only line through P and Q}\end{array}\right\}\xrightarrow{⑤}$$

$$\left.\begin{array}{l}\xrightarrow{⑤} \text{l and m are the same line}\\ \text{⑥ but l and m are different lines is true}\end{array}\right\}\xrightarrow{⑦}$$

$\xrightarrow{⑦}$ l and m intersect in a second point is false

∴ P is the only point of intersection

The reason for ② and ④ is Postulate 4, "For any two different points, there is exactly one line containing both points".

The theorem we have proved is a uniqueness theorem. Existence is established in the hypothesis. Uniqueness is generally proved by an indirect proof.

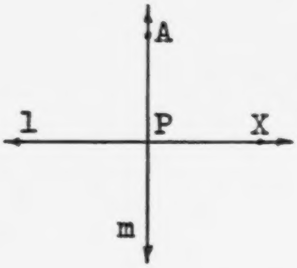
The following is a major theorem of existence and uniqueness that requires this method. We'll consider it with an informal paragraph proof.

Theorem 10-1: In a plane, at a given point on a line, there is exactly one line perpendicular to the given line.

Given: P and X are on l in a given plane

Prove: (1) There exists a line, m, in the given plane, perpendicular to l through P

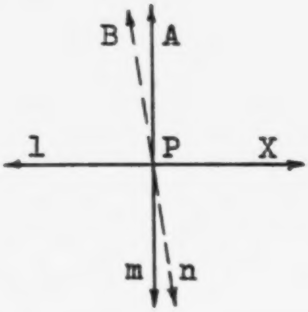
(2) m is the only line in the given plane perpendicular to l through P



Argument

Part (1): In one half-plane of the given plane, there is exactly one ray, $\overrightarrow{PA}$, such that $m\angle APX = 90$. Therefore, $\angle APX$ is a right angle and $m \perp l$.

Part (2): Suppose there is a second line, n, in the given plane and perpendicular to l through P. Then $\angle BPX$ is a right angle and $m\angle BPX = 90$. But $m\angle APX = 90$ so $\overrightarrow{PB}$ and $\overrightarrow{PA}$ both correspond to 90. This contradicts theorem 9-2 which states that there is exactly one such ray. Therefore, the supposition is false and m is the only line in the given plane perpendicular to l through P.



Indirect proof is not frequently used in our course, but there will be more cases in which it will be necessary.

EXERCISE SET 10.1

Write the contrapositive of each of the following.

1. If a song is not a hit, then it will not sell a million records.
2. If two angles are not congruent, then they are not vertical angles.
3. If two positive integers are odd, then their sum is even.
4. If the diagonals of a quadrilateral are not congruent, then it is not a rectangle.

Using the word "suppose", write the first statement that would be made in an indirect proof of each of the following.

5. If $x + 5 = 7$, then $x \neq 3$.

6. If two coplanar lines are perpendicular to the same line, then they do not intersect.

7. If the square of a number is 2, it is not rational.

8. If a point is not on a line, there is not more than one line through that point perpendicular to the given line.

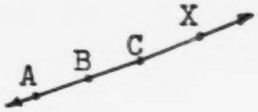
Even numbers can be represented in the form $2n$ where n is any integer, and odd numbers in the form $2n + 1$ where n is any integer.

Show an indirect proof for each of the following.

9. Prove: If a prime number is greater than 2, then it is odd.

10. Prove: If two complementary angles are congruent, then the measure of each is 45.

11. Prove: If A, B, and C are collinear and B, C, and X are not collinear, then A, C, and X are not collinear.



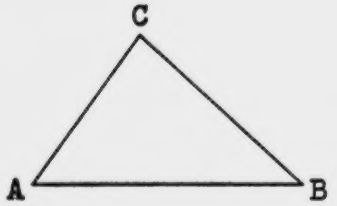
12. Remember Uncle Casper's haunted house!
Prove: If a house has 5 rooms and 33 windows, at least one room has an odd number of windows.

CHAPTER 11 - CONGRUENCE

or HAVEN'T I SEEN YOU BEFORE

Definitions of Triangles

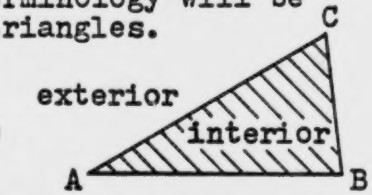
This chapter is liable to give you a strong case of triangleitis. The best cure is to know everything about these three-faced creatures and to be aware of their future consequences.



A *triangle* is the union of three *segments joining* three noncollinear points.

In the triangle shown, each of the three points, A, B, and C, is called a *vertex*. The plural of vertex is *vertices*. A triangle is named by *its vertices*, ΔABC.

A triangle is said to have six parts, three sides and three angles. A side is *included* between two angles *if its endpoints are the vertices of* the two angles. $\overline{AB}$ is included between $\angle A$ and $\angle B$. An angle is *included* between two sides if its *sides contain* those two *sides.* $\angle A$ is included between $\overline{AB}$ and $\overline{AC}$. $\overline{AB}$ is said to be *opposite* $\angle C$. $\angle A$ is said to be *opposite* $\overline{BC}$. This terminology *will be* frequently used *in our discussion* of triangles.



A triangle separates a plane into three subsets, the interior of the triangle, the triangle itself, and the exterior of the triangle.

The *interior* of a triangle is the *intersection of the* interiors of its three angles. The exterior is the set of all points in the plane that do not belong to the triangle or to its interior. Notice in particular that the points in the interior of a triangle do not belong to the triangle itself.

Triangles are classified and named according to the relationships of their sides and their angles. You are probably already familiar with the following definitions.

A *scalene triangle* is a triangle with no two sides congruent.

An *isosceles triangle* is a triangle with at least two sides congruent.

An *equilateral triangle* is a triangle with three sides congruent.

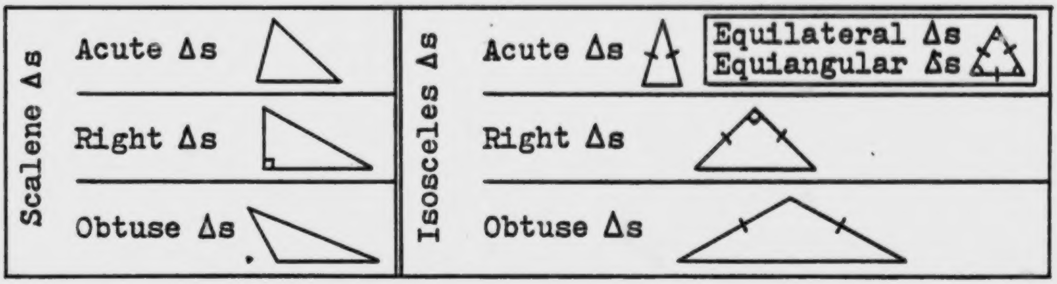
An *equiangular triangle* is a triangle with three angles congruent.

An *acute triangle* is a triangle with three acute angles.

A *right triangle* is a triangle with a right angle.

An *obtuse triangle* is a triangle with an obtuse angle.

The following modified Venn diagram shows that every triangle is either scalene or isosceles and simultaneously acute, right, or obtuse.

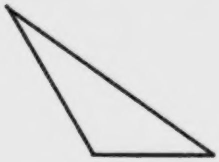


By definition, every equilateral triangle is isosceles. We will show later that the set of equilateral triangles and the set of equiangular triangles are the same set. The marks on the sides of the isosceles triangles indicate the congruent sides, and right angles are indicated as shown in the right triangles. These are convenient designations for relations or properties that cannot be assumed from a figure alone.

Congruent Triangles and Correspondence

Congruence can be thought of in a physical sense as referring to objects having the same shape and size. Any set of objects coming off an assembly line can be thought of as being a congruent set of objects. Our definitions for congruent segments and congruent angles are consistent with this idea.

The figures shown here illustrate congruent triangles.

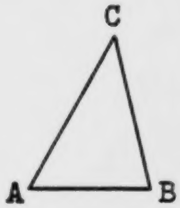


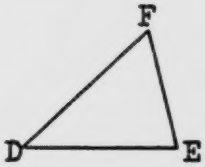
They do have the same shape and size even though they are not in the same relative position. You may even have to flip one over to show that it is actually a "twin" to another one. But our abstract triangles cannot be moved or flipped and in order to state a definition for congruent triangles, we establish the idea of corresponding parts.

Since all triangles have three vertices, we can establish a one to one correspondence between the vertices of any two triangles. This can be done in six different ways.

A c D is read "A corresponds to D".

A c D, B c E, C c F
A c D, B c F, C c E
A c E, B c D, C c F
A c E, B c F, C c D
A c F, B c D, C c E
A c F, B c E, C c D





Each of these may be abbreviated. The first correspondence may be written ABC c DEF, in which corresponding vertices are written in the same order. For example, CAB c FDE represents the same correspondence.

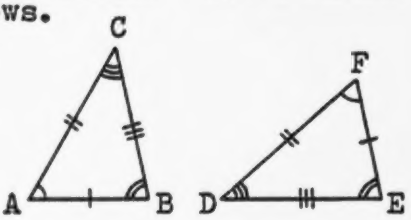
For a particular correspondence of vertices, we can identify the corresponding sides and angles. If ABC c FED, we have $\overline{AB}$ c $\overline{FE}$, $\overline{AC}$ c $\overline{FD}$, $\overline{BC}$ c $\overline{ED}$, ∠A c ∠F, ∠B c ∠E, ∠C c ∠D. We don't need a figure to do this. We simply take them in the same order they are written in the correspondence.

We can now define congruent triangles.

If, for some correspondence of vertices of two triangles, the sides and angles of one triangle are congruent to the corresponding sides and angles of the other, the correspondence is a congruence, and each triangle is congruent to the other.

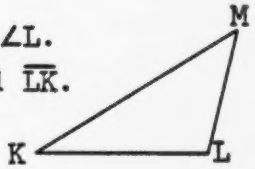
When we write $\triangle ABC \cong \triangle FED$, we are stating $ABC \leftrightarrow FED$ is a congruence. It is not sufficient to merely name the triangles. We must write their vertices in the order of correspondence. $\triangle ABC \cong \triangle DEF$ is a different statement. A figure is not necessary to identify corresponding parts, but it is convenient to mark them as shown. The definition is expressed symbolically as follows.

$$\triangle ABC \cong \triangle FED \longleftrightarrow \left\{\begin{array}{l} \overline{AB} \cong \overline{FE} \\ \overline{AC} \cong \overline{FD} \\ \overline{BC} \cong \overline{ED} \\ \angle A \cong \angle F \\ \angle B \cong \angle E \\ \angle C \cong \angle D \end{array}\right.$$



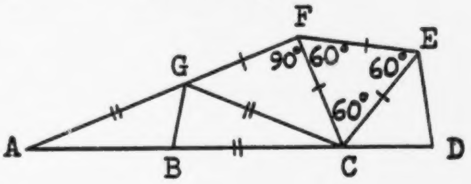
EXERCISE SET 11.1

1. Name the six parts of $\triangle KLM$.
2. Name the side included between $\angle M$ and $\angle L$.
3. Name the angle included between $\overline{KM}$ and $\overline{LK}$.
4. Name the side opposite $\angle K$.
5. Name the angle opposite $\overline{KM}$.
6. Is M in the interior of the triangle? the exterior?



In the figure shown, congruent segments are marked appropriately. Segments not marked are not congruent to any other segments. Name each of the following.

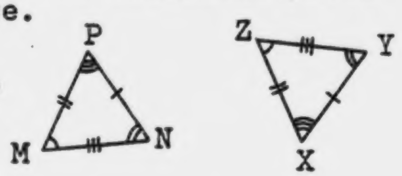
7. An obtuse scalene triangle
8. A right isosceles triangle
9. An equilateral triangle
10. An acute scalene triangle
11. An obtuse isosceles triangle
12. A right scalene triangle
13. An equiangular triangle
14. An acute isosceles triangle that is not equilateral



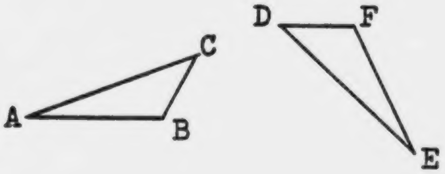
15. If △COR ≅ △AFD, name the six pairs of congruent parts according to this correspondence.

16. If the triangles shown are congruent by the correspondence indicated by the markings, complete the following.

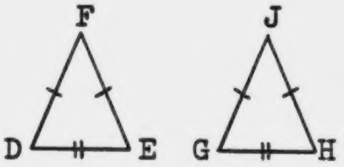
△MNP ≅ △ ?



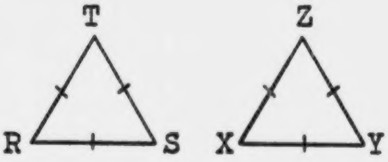
17. Using the appearance of the triangles shown, make a statement of congruence.



18. Isosceles triangles may be congruent by two different correspondences. For the triangles shown, write two different congruent triangle statements.

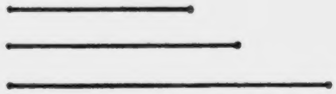


19. Equilateral triangles may be congruent by six different correspondences. For the triangles shown, write all six congruent triangle statements.



Congruence Postulates

You can probably guess that we don't need to show that all six corresponding parts of two triangles are congruent in order to show that the triangles are congruent. In fact, we can do it with three of the six if they are the right three.

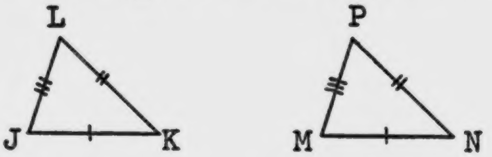


Consider the three segments shown. Construct a triangle whose sides are congruent to the three segments. Do you think you could construct a second triangle with the same segments that would not be a twin to the first one? Try it. Actually, if ten different people were to construct their own triangle with these segments, you would find all ten of them to be congruent to each other. That is, the corresponding angles would be congruent as well as the sides. This leads us to our first congruence postulate. We'll state it formally first as a correspondence and then in a shorter form that we'll use for later postulates.

Postulate 10: If a correspondence between the vertices of two triangles is such that the three sides of one are congruent to the three sides of the other, then that correspondence is a congruence.

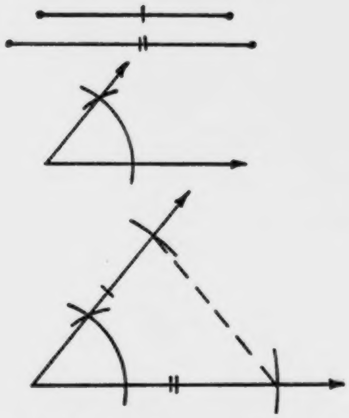
If three sides of one triangle are congruent to three sides of another triangle, the triangles are congruent. (SSS Postulate)

$$\left.\begin{array}{l}\overline{JK}\cong\overline{MN}\\ \overline{KL}\cong\overline{NP}\\ \overline{LJ}\cong\overline{PM}\end{array}\right\}\longrightarrow \triangle JKL\cong\triangle MNP$$



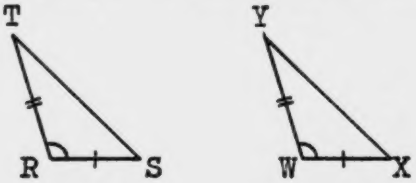
Similar experiments could be done for other cases. Let the two segments shown be congruent to two sides of a triangle and let the angle be congruent to the included angle between them.

As we can see in our construction, the third side has no choice but to be the dashed line, and we can say that the triangle is determined with respect to shape and size. That is, all other triangles with the same conditions would be congruent to this one.



Postulate 11: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the triangles are congruent. (SAS Postulate)

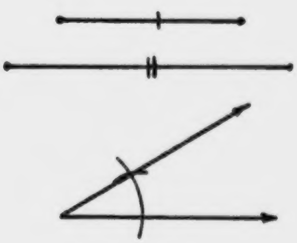
$$\left.\begin{array}{l}\overline{RS}\cong\overline{WX}\\ \angle R\cong\angle W\\ \overline{RT}\cong\overline{WY}\end{array}\right\}\longrightarrow \triangle RST\cong\triangle WXY$$

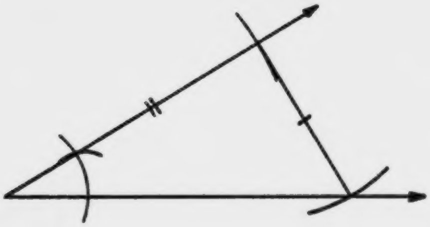


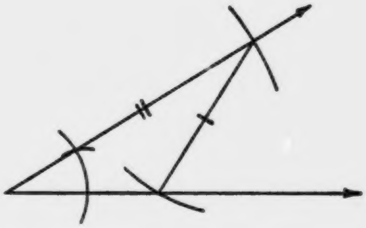
We write SAS to indicate the fact that the angle is the included angle between the two sides.

Do you think there would be a SSA postulate where the angle is not the included angle? The best way to answer the question is to experiment.

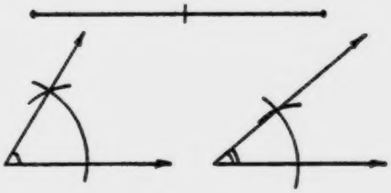
Use the segments shown and let the angle be an unincluded angle between the two sides. Your construction may look like the first one shown here. But your friend, Sammy, satisfied the same conditions with the second figure. Obviously, a triangle is not determined, and there is no SSA postulate for congruent triangles. This is important.

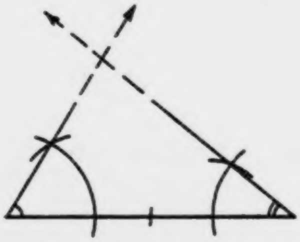






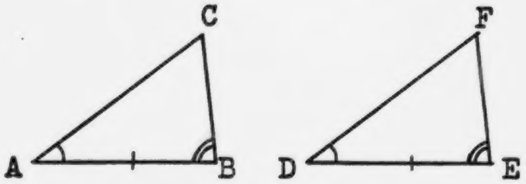
The next logical experiment would involve two angles and a side. Let the segment shown be congruent to the included side between the two angles given. The construction shows clearly that the third vertex is determined by the intersection of two rays. This gives us our third congruence postulate.





Postulate 12: If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, the triangles are congruent. (ASA Postulate)

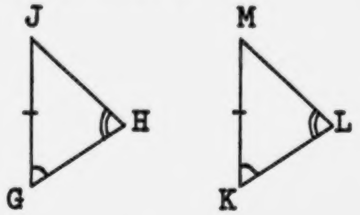
$$\left.\begin{array}{l}\angle A \cong \angle D\\ \overline{AB} \cong \overline{DE}\\ \angle B \cong \angle E\end{array}\right\}\longrightarrow \triangle ABC \cong \triangle DEF$$



Of course, you're already asking "What about an AAS postulate?". Actually, the AAS correspondence can be proved to be a congruence as a theorem, but we don't have the reasons available yet to do this. We'll state it now as postulate 12a, and then restate it as a theorem after we prove it later.

Postulate 12a: If two angles and an unincluded side of one triangle are congruent to the corresponding parts of another triangle, the triangles are congruent. (AAS Postulate)

$$\left.\begin{array}{l}\angle G \cong \angle K\\ \angle H \cong \angle L\\ \overline{GJ} \cong \overline{KM}\end{array}\right\}\longrightarrow \triangle GHJ \cong \triangle KLM$$



Notice that the unincluded side must be the corresponding unincluded side. Otherwise the correspondence would not be a congruence.

We now have four ways to prove triangles congruent without proving all six corresponding parts congruent. They are very valuable tools in geometry.

The proofs of the three theorems that follow will be left as exercises.

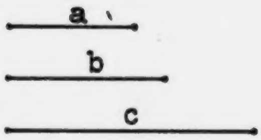
Theorem 11-1: Congruence of triangles is reflexive.

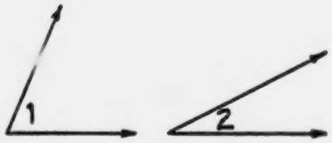
Theorem 11-2: Congruence of triangles is symmetric.

Theorem 11-3: Congruence of triangles is transitive.

EXERCISE SET 11.2

Copy the segments and angles shown. Construct triangles illustrating the postulates of this section in the following exercises, and observe that the triangle is determined in each case.

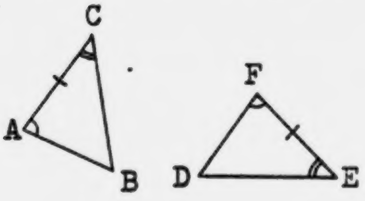




1. SSS using a, b, and c
2. SAS using b, c, and ∠1
3. ASA using c, ∠1, and ∠2
4. AAS using ∠1, ∠2, and b
5. Is it possible for two triangles to be congruent without at least one side of one triangle congruent to a side of the other?

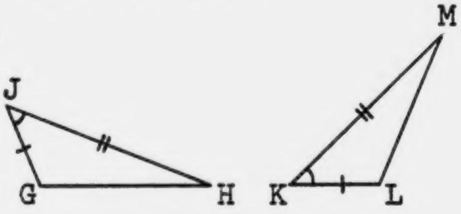
For each figure complete the statement of congruence and write the postulate that justifies it.

6.



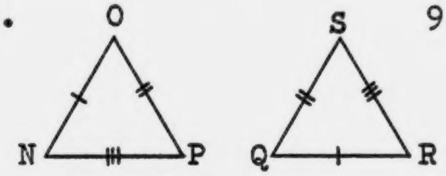
ΔABC ≅ Δ ?

7.



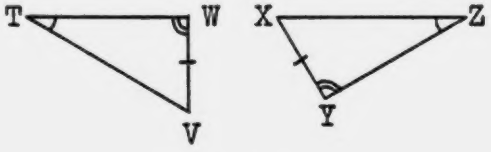
ΔGHJ ≅ Δ ?

8.



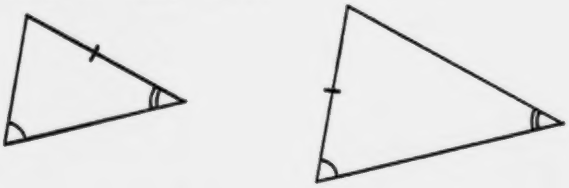
ΔNPO ≅ Δ ?

9.



ΔTVW ≅ Δ ?

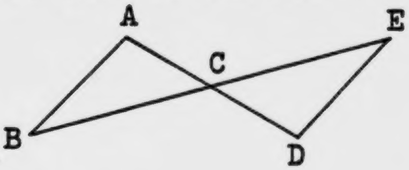
10. The following two triangles have two angles and an unincluded side of one congruent to two angles and an unincluded side of the other. Must they be congruent? Why or why not?



11. Prove theorem 11-1.
12. Prove theorem 11-2.
13. Prove theorem 11-3.

The Use of a Figure in a Proof

Since any figure is merely a representation of abstract elements that we cannot really see in the physical sense, a figure is not actually necessary to a geometry proof. But even the simplest proofs would be much more lengthy and awkward without a figure, and we use it as a powerful aid to help us through. When we state the hypothesis as "given" in a proof, we are automatically including the figure and any relationships that are acceptable on the basis of the figure. So we must know what we can and cannot accept from a figure alone.

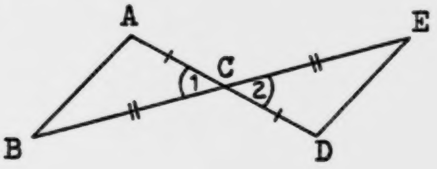


For the figure shown here, we accept the fact that A, C, and D are collinear, B, C, and E are collinear, AD and BE intersect at C, C is between B and E, C is between A and D, and there are two pairs of vertical angles at C.

We cannot accept any relationships about congruent segments or angles regardless of appearance. If such relationships do exist for a particular problem, they must be stated as given or they must be proved. We'll use this figure as an example of a simple proof.

Given: the figure shown with
$\overline{AC} \cong \overline{DC}$ and $\overline{BC} \cong \overline{EC}$

Prove: $\triangle ABC \cong \triangle DEC$



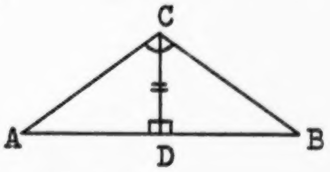
Argument

$$\left.\begin{array}{l} \textcircled{1}\ \overline{AC} \cong \overline{DC} \\ \textcircled{2}\ \overline{BC} \cong \overline{EC} \\ \textcircled{3}\ \angle 1 \cong \angle 2 \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle ABC \cong \triangle DEC$$

① Given
② Given
③ Vert. ∠s are ≅
④ SAS Postulate

Marking the corresponding parts as shown makes it very easy to see which congruence postulate justifies the last step.

The perpendicular relation is another one that cannot be accepted from a figure alone.



Given: $\overline{CD} \perp \overline{AB}$, $\angle ACD \cong \angle BCD$

Prove: $\triangle ADC \cong \triangle BDC$

Argument

$$\text{①}\ \overline{CD}\perp\overline{AB} \xrightarrow{\text{②}} \left.\begin{array}{r} \angle ADC \cong \angle BDC \\ \text{③}\ \angle ACD \cong \angle BCD \\ \text{④}\ \overline{CD} \cong \overline{CD} \end{array}\right\} \xrightarrow{\text{⑤}} \triangle ADC \cong \triangle BDC$$

① Given
② If 2 lines are ⊥, the ∠s in a linear pair are ≅.
③ Given
④ Reflexive prop. of ≅ segments
⑤ ASA Postulate

This example brings up several points worth mentioning. We begin to make use of previously proved theorems, such as theorem 9-12 (② above), as reasons in a later proof. In this case, we cut out all statements about right angles or measures of angles in going directly to statement ②. Notice that given statements are not always made first in a proof. We write them where we need them in the logical structure of the proof. An important point about proving triangles congruent is that we can never have fewer than three statements in the braces to the left of the congruent triangles statement. You may feel that the statement $\overline{CD} \cong \overline{CD}$ is trivial and unnecessary, but it is actually quite necessary as the third statement required for congruence. We are really saying that $\overline{CD}$ of $\triangle ACD$ is congruent to $\overline{CD}$ of $\triangle BCD$. The reflexive property of congruent angles will also be used when an angle is common to two triangles.

The Word RIGOR in Mathematics

Before looking at more involved proofs, we need to discuss a point related to our approach to proof. The word "rigor" in mathematics refers to the degree of srtictness required in making statements and reasons in the argument of a proof. In the strictest sense, every logical statement should be made in the argument such that there is exactly one reason for each statement. But the more experienced we become, the more burdensome this is.

We can begin to see ahead two or three steps and feel that it's a nuisance to write the intermediate steps. But this can lead to problems also. Arguing a proof in geometry is something like a lawyer presenting a case to a jury. If he reaches conclusions without substantial justification, the jury (your teacher) will probably think he's faking it. And sometimes he (the student) is. It is also possible to go too far the other way. Some students make many more statements in the argument than are necessary hoping that the right ones will be among them. This is similar to the lawyer who presents irrelevant material in a trial. His statements may be true, but they would weaken his case because the jury recognizes that he doesn't really comprehend what he is trying to prove. We would like to find a happy medium that would allow us to present a convincing argument without too many nuisance steps. But this varies considerably from one individual to another, so it's not easy.

We will begin by allowing the statements in a biconditional definition to be interchangeable. We have referred to these statements as equivalent. For example, the definition of congruent segments says $AB = CD \longleftrightarrow \overline{AB} \cong \overline{CD}$. We will allow either of these to be written in place of the other without writing both. Some other examples of equivalent statements based on definitions are as follows.

$m\angle ABC = m\angle DEF \longleftrightarrow \angle ABC \cong \angle DEF$

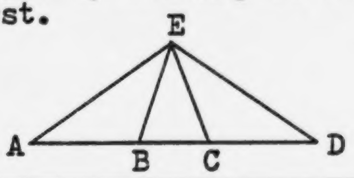
$\overrightarrow{AB} \perp \overrightarrow{AC} \longleftrightarrow \angle BAC$ is a rt. $\angle$

$m\angle A + m\angle B = 180 \longleftrightarrow \angle A$ suppl. $\angle B$

$\angle A$ is an acute $\angle \longleftrightarrow m\angle A < 90$

$\overline{CD}$ is bisected at $X \longleftrightarrow CX = XD$

Any other "agreements" as to what may or may not be omitted as you go through the course is between you and your teacher. We'll illustrate the point by showing an example with all necessary statements first.



Given: $\overline{AB} \cong \overline{DC}$, $\overline{AE} \cong \overline{DE}$, $\angle A \cong \angle D$

Prove: $\triangle ACE \cong \triangle DBE$

Argument

$$\textcircled{1}\ \overline{AB} \cong \overline{DC} \xrightarrow{\textcircled{2}} AB = DC \xrightarrow{\textcircled{3}} \left.\begin{array}{r} AB + BC = DC + BC \\ \textcircled{4}\ AB + BC = AC \\ \textcircled{5}\ DC + CB = DB \end{array}\right\} \xrightarrow{\textcircled{6}} AC = DB \xrightarrow{\textcircled{7}}$$

$$\xrightarrow{\textcircled{7}} \left.\begin{array}{r} \overline{AC} \cong \overline{DB} \\ \textcircled{8}\ \overline{AE} \cong \overline{DE} \\ \textcircled{9}\ \angle A \cong \angle D \end{array}\right\} \xrightarrow{\textcircled{10}} \triangle ACE \cong \triangle DBE$$

You may feel that statements ② through ⑥ are nuisance steps and that $\overline{AB} \cong \overline{DC} \longrightarrow \overline{AC} \cong \overline{DB}$ is obvious. The fact is, we've already proved ② through ⑥ on page 66, but as a problem rather than a formal theorem. If you wish, you may state it as a theorem (even name it after yourself and be famous). Then our proof would look like this.

$$\text{①}\ \overline{AB} \cong \overline{DC} \xrightarrow{\text{②}} \left.\begin{array}{l} \overline{AC} \cong \overline{DB} \\ \text{③}\ \overline{AE} \cong \overline{DE} \\ \text{④}\ \angle A \cong \angle D \end{array}\right\} \xrightarrow{\text{⑤}} \triangle ACE \cong \triangle DBE$$

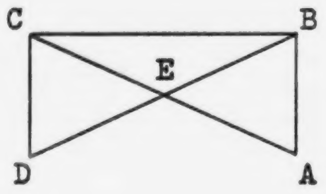
① Given
② Andy's theorem
③ ④ Given
⑤ SAS

In any case, make sure you can justify every statement in the argument of a proof by an acceptable reason.

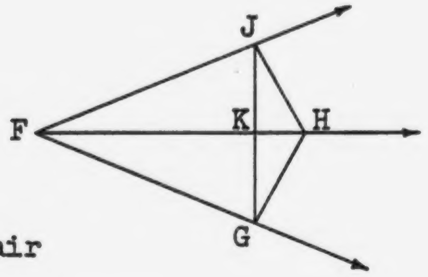
EXERCISE SET 11.3

In exercises 1-14, tell whether the statement may or may not be accepted on the basis of the figure. (yes or no)

1. E is between A and C
2. $\overline{AB} \perp \overline{BC}$
3. $\angle BAE \cong \angle CDE$
4. E is the midpoint of $\overline{AC}$
5. $\angle AEB$ and $\angle DEC$ are vertical $\angle$s
6. $\overline{AB} \cong \overline{DC}$
7. $\angle DEC$ and $\angle CEB$ are a linear pair

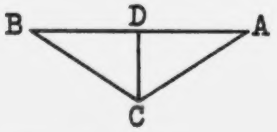


8. $\overline{FH} \perp \overline{JG}$
9. $FG = FJ$
10. $\overrightarrow{FH}$ bisects $\angle JFG$
11. $\angle JKF \cong \angle GKF$
12. $\overrightarrow{FH}$ is between $\overrightarrow{FG}$ and $\overrightarrow{FJ}$
13. $\angle JKF$ and $\angle GKF$ are a linear pair
14. $\triangle GHJ$ is isosceles



Copy and complete each proof. Mark the appropriate segments and angles on the figure and write all reasons.

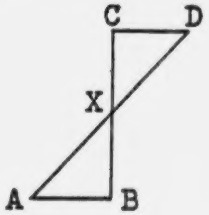
15. Given: $\angle A \cong \angle B$, $\overline{CD}$ bisects $\angle ACB$
Prove: $\triangle ACD \cong \triangle\ ?$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ \angle A \cong \angle B \\ \textcircled{2}\ \angle ? \cong \angle ? \\ \textcircled{3}\ \ ? \cong ? \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle ACD \cong \triangle\ ?$$

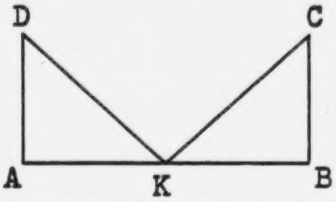
16. Given: $\overline{AD}$ and $\overline{BC}$ bisect each other
Prove: $\triangle ABX \cong \triangle\ ?$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ \ ? \cong ? \\ \textcircled{2}\ \ ? \cong ? \\ \textcircled{3}\ \angle ? \cong \angle ? \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle ABX \cong \triangle\ ?$$

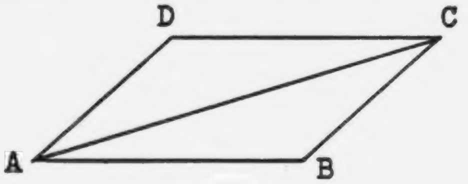
17. Given: K is midpoint of $\overline{AB}$
$AD = BC$, $KD = KC$
Prove: $\triangle KAD \cong \triangle\ ?$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ \overline{AD} \cong \overline{BC} \\ \textcircled{2}\ \overline{KD} \cong \overline{KC} \\ \textcircled{3}\ \ ? \cong ? \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle KAD \cong \triangle\ ?$$

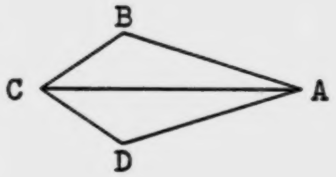
18. Given: $\angle BCA \cong \angle DAC$
$\angle BAC \cong \angle DCA$
Prove: $\triangle ABC \cong \triangle\ ?$



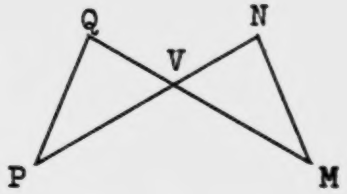
Argument

$$\left.\begin{array}{l} \textcircled{1}\ \angle BCA \cong \angle DAC \\ \textcircled{2}\ \angle BAC \cong \angle DCA \\ \textcircled{3}\ \ \ ? \cong ? \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle ABC \cong \triangle\ ?$$

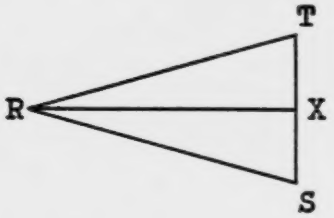
19. Given $\overline{AB} \cong \overline{AD}$, $\overline{CB} \cong \overline{CD}$
    Prove: $\triangle ABC \cong \triangle ADC$



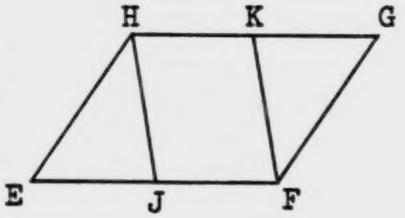
20. Given: $\overline{PQ} \perp \overline{QM}$, $\overline{MN} \perp \overline{NP}$,
    $\overline{QV} \cong \overline{NV}$
    Prove: $\triangle PQV \cong \triangle MNV$



21. Given: $\overline{RX}$ bisects $\angle TRS$
    $\overline{RX} \perp \overline{ST}$
    Prove: $\triangle RSX \cong \triangle RTX$



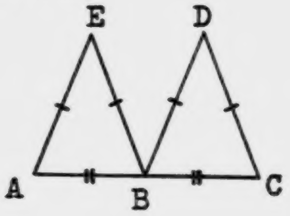
22. Given: $EF = GH$, $\overline{HE} \cong \overline{FG}$,
    $\angle E \cong \angle G$
    K is midpoint of $\overline{HG}$
    J is midpoint of $\overline{EF}$
    Prove: $\triangle EJH \cong \triangle GKF$



With the segments marked as shown, make the statements necessary to prove each of the following.

23. $\triangle ABE \cong \triangle BCD$

24. $\triangle ABE \cong \triangle CBD$



Analysis of a Proof

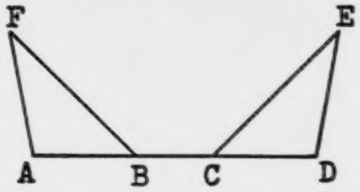
The example in the previous section involved overlapping triangles. As the figures become more involved, the proofs may become more difficult. Unfortunately, there is no specified set of steps to take in proving a theorem as there may be in solving a particular kind of equation. But there are helpful hints that can aid us in attacking a proof where a plan is not immediately obvious.

First, always have scratch paper handy. If you see a possible start, but you're not sure where it would lead, go ahead and try it. It may lead to a dead end, but nothing is really lost. It's all part of the exploration process.

The most common approach to a difficult proof is an analysis in which you essentially work backwards from the statement you are trying to prove. The reasoning would be "I can prove this if I could prove that first". Then we look for ways to prove the new statement until we eventually get a connection with something that we know is true. The best assurance of making this connection is complete familiarity with all definitions, postulates, and previously proved theorems that may have some bearing on this particular proof. Without this knowledge, proof becomes impossible. The following is a fairly simple example of an analysis, but the general idea applies to more involved proofs.

Given: $\overline{AB} \cong \overline{DC}$, $\overline{FB} \cong \overline{EC}$,
$\angle FBC \cong \angle ECB$

Prove: $\triangle ABF \cong \triangle DCE$



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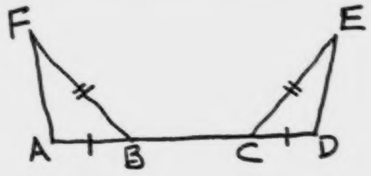
scratch paper

We mark the corresponding parts of the triangles that we know are congruent, and we know that the end of our proof will look like this.



$$\left.\begin{array}{r}\overline{AB} \overset{?}{\cong} \overline{DC}\\ \overline{FB} \cong \overline{EC}\end{array}\right\}\longrightarrow \triangle ABF \cong \triangle DCE$$

We either need the third sides congruent to use SSS or the included angles congruent to use SAS. The angles look like the best bet because of the other given statement.

$$? \longrightarrow \left.\begin{array}{r}\angle ABF \cong \angle DCE\\ \overline{AB} \cong \overline{DC}\\ \overline{FB} \cong \overline{EC}\end{array}\right\}\longrightarrow \triangle ABF \cong \triangle DCE$$

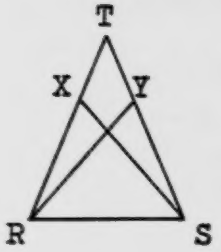
Now we need a connecting link with that other given statement. A connection may seem reasonable to you, but unless you recall (or review) theorems 9-4 and 9-6, you may flounder with it for some time. Look them up now and see if you can finish off the proof.

Figures with Overlapping Triangles

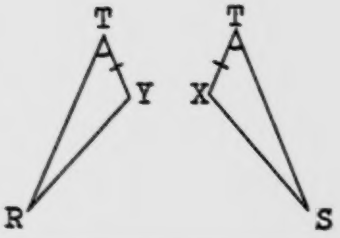
In the figure shown below, you should see three pairs of triangles with congruence possibilities. Some students find it helpful to make a new drawing by separating the triangles you are interested in as we will do here. Others have success by using different colored pencils and tracing the desired triangles with two colors. It not only looks more artistic, but it impresses your friends. With experience, you may not need to do either.

Given: $\overline{RX} \cong \overline{SY}$, $\overline{XT} \cong \overline{YT}$

Prove: $\Delta RYT \cong \Delta SXT$



If we separate the triangles and mark what we know, we see this new figure. An analysis should then lead you to the following proof.



Argument

$$\left.\begin{array}{l}\text{①}\ RX = SY\\ \text{②}\ XT = YT\end{array}\right\}\xrightarrow{\text{③}} RX + XT = SY + YT$$

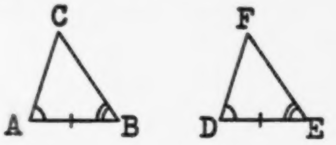
$$\left.\begin{array}{r}RX + XT = SY + YT\\ \text{④}\ RX + XT = RT\\ \text{⑤}\ SY + YT = ST\end{array}\right\}\xrightarrow{\text{⑥}} \overline{RT} \cong \overline{ST}$$

$$\left.\begin{array}{r}\overline{RT} \cong \overline{ST}\\ \text{②}\ \overline{YT} \cong \overline{XT}\\ \text{⑦}\ \angle RTY \cong \angle STX\end{array}\right\}\xrightarrow{\text{⑧}} \Delta RYT \cong \Delta SXT$$

You supply the reasons.

Congruent triangles are frequently used to prove certain segments or angles congruent. Once we have proved two triangles congruent, the definition of congruent triangles tells us that the other three corresponding parts are also congruent.

$$\left.\begin{array}{l} \textcircled{1}\ \angle A \cong \angle D \\ \textcircled{2}\ \overline{AB} \cong \overline{DE} \\ \textcircled{3}\ \angle B \cong \angle E \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle ABC \cong \triangle DEF \begin{array}{l} \xrightarrow{\textcircled{5}} \overline{AC} \cong \overline{DF} \\ \xrightarrow{\textcircled{6}} \angle C \cong \angle F \\ \xrightarrow{\textcircled{7}} \overline{BC} \cong \overline{EF} \end{array}$$



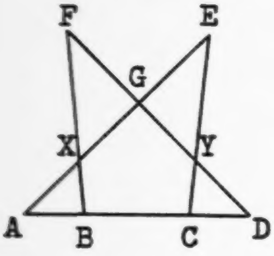
Statement ④ implies any one or all of ⑤, ⑥, and ⑦ depending on what a particular problem requires. The reason is, of course, the definition of congruent triangles, but we will write it as follows.

⑤ ⑥ ⑦ Corresponding parts of congruent triangles are congruent.
(Abbreviation) Corr. parts of ≅ $\triangle$s are ≅.

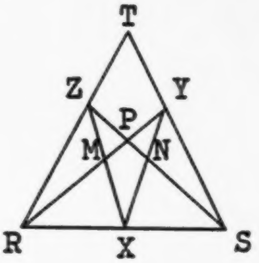
EXERCISE SET 11.4

From the figures shown, name all pairs of "possible" congruent triangles you can find based on appearance.

1.

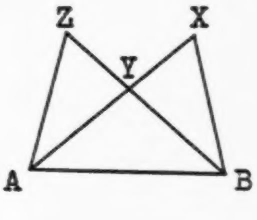


2.



In the following proofs, if you cannot see a straightforward approach, write the conclusion and try to work backwards.

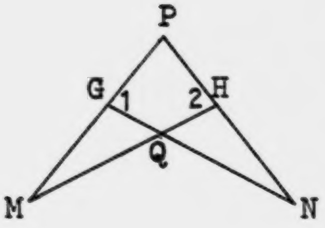
3. Given: $\overline{ZA} \cong \overline{XB}$
$\angle ZAB \cong \angle XBA$
Prove: $\triangle ABZ \cong \triangle BAX$



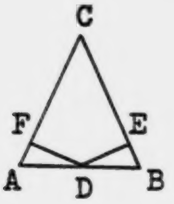
4. Given: $\angle X \cong \angle Z$
$\overline{XY} \cong \overline{ZY}$
Prove: $\triangle BXY \cong \triangle AZY$

5. Given: $\overline{GP} \cong \overline{HP}$, $\angle 1 \cong \angle 2$
Prove: $\triangle PNG \cong \triangle PMH$

6. Given: $\angle 1 \cong \angle 2$, $MG = NH$
Prove: $\triangle GMQ \cong \triangle HNQ$



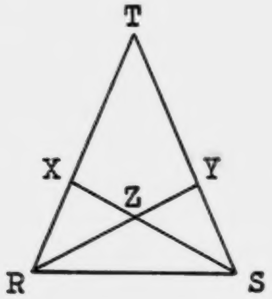
7. Given: $\overline{DF} \perp \overline{AC}$, $\overline{DE} \perp \overline{BC}$
$\overline{AF} \cong \overline{BE}$, $\angle A \cong \angle B$
Prove: $\overline{AD} \cong \overline{BD}$



8. Given: $\overline{RT} \cong \overline{ST}$, $\overline{TY} \cong \overline{TX}$
Prove: $\overline{RY} \cong \overline{SX}$

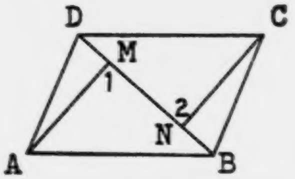
9. Given: $\overline{RX} \cong \overline{SY}$, $\overline{RY} \cong \overline{SX}$
Prove: $\angle RSX \cong \angle SRY$

10. Given: $\overline{RZ} \cong \overline{SZ}$, $\overline{XZ} \cong \overline{YZ}$
Prove: $\overline{RX} \cong \overline{SY}$



11. Sometimes it is necessary to prove more than one pair of triangles congruent. Complete the following proof.

Given: $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$, $\angle 1 \cong \angle 2$
Prove: $\overline{AM} \cong \overline{CN}$

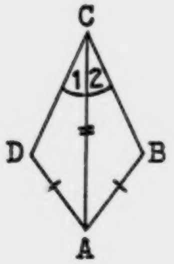


Argument

$$\left.\begin{array}{l}\textcircled{1}\ \overline{AB}\cong\overline{CD}\\ \textcircled{2}\ \overline{AD}\cong\overline{CB}\\ \textcircled{3}\ \ ?\cong ?\end{array}\right\}\xrightarrow{\textcircled{4}}\triangle ABD\cong\triangle CDB\xrightarrow{\textcircled{5}}\left.\begin{array}{r}\angle ?\cong\angle ?\\ \textcircled{6}\ \angle ?\cong\angle ?\\ \textcircled{1}\ \ ?\cong\ ?\end{array}\right\}\xrightarrow{\textcircled{7}}\triangle ?\cong\triangle ?\xrightarrow{\textcircled{8}} ?\cong ?$$

12. It is given that $\overline{AD} \cong \overline{AB}$ and $\angle 1 \cong \angle 2$. Using these together with $\overline{AC} \cong \overline{AC}$, is this sufficient to prove $\overline{CD} \cong \overline{CB}$?

    Draw another figure satisfying the same given statements in which it is obvious that $\overline{CD} \ncong \overline{CB}$.



Introducing Auxiliary Lines

There are times when a figure for a particular theorem does not allow us to make the statements necessary to prove the theorem. It may be possible to add more lines, rays, or segments to the given figure so that it has the relationships and properties that we can make statements about. These are called auxiliary lines, rays, or segments. But we must always be careful to state the specific conditions we are placing on this new element, and we must be able to justify its existence and uniqueness. This is where the earlier postulates and theorems about existence and uniqueness are important. We will restate the ones that will be used for this purpose and illustrate their use.

Postulate 4: Two different points determine a line.

Theorem 7-1: On a ray, there is exactly one point at a given distance from the endpoint of the ray. (Point plotting theorem)

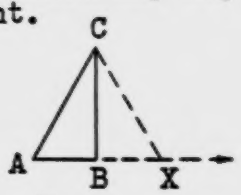
Theorem 7-2: A segment has exactly one midpoint.

Theorem 9-2: Through the endpoint of a ray lying in the edge of a half-plane, there is exactly one other ray in the half-plane such that the measure of the angle formed by the two rays is a given number between 0 and 180. (Angle construction theorem)

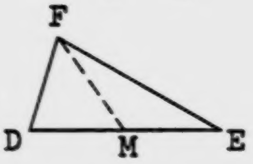
Theorem 9-3: An angle has exactly one bisector.

Theorem 10-1: In a plane, at a given point on a line, there is exactly one line perpendicular to the given line.

Auxiliary lines are generally shown with a dash line to distinguish them from the basic figure. We allow any ray or segment to be extended beyond its endpoint.



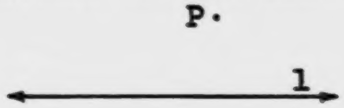
Given $\triangle ABC$, extend $\overline{AB}$ beyond B.
Let X be a point such that $BX = BA$.
Reason: Point plotting theorem
Draw $\overline{CX}$
Reason: Two points determine a line.



Given $\triangle DEF$
Let M be the midpoint of $\overline{DE}$.
Reason: A segment has exactly one midpoint.
Draw $\overline{FM}$
Reason: Two points determine a line.

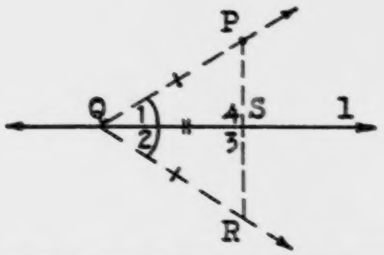
Now we can consider an important existence and uniqueness theorem about perpendiculars.

Theorem 11-4: Through a point outside a line, there is exactly one line perpendicular to that line.



Once again we must show two things.
(1) Such a perpendicular does exist.
(2) It is the only one.

(1) Given: Point P not on line l
Prove: A line containing P and $\perp$ l does exist



Argument

① Let Q be any point on l. Draw $\overrightarrow{QP}$.
② Construct $\angle 2 \cong \angle 1$ on opposite side of l.
③ Let R be a point as shown such that $QR = QP$.
④ Draw $\overline{PR}$ intersecting l at S.

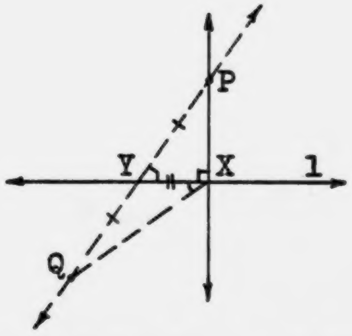
$$\left.\begin{array}{l} \text{②}\ \angle 2 \cong \angle 1 \\ \text{③}\ \overline{QR} \cong \overline{QP} \\ \text{⑤}\ \overline{QS} \cong \overline{QS} \end{array}\right\} \xrightarrow{\text{⑥}} \triangle RQS \cong \triangle PQS \xrightarrow{\text{⑦}} \angle 3 \cong \angle 4 \xrightarrow{\text{⑧}} \overline{PR} \perp l$$

$\therefore$ A line $\overleftrightarrow{PR}$ does exist $\perp$ l.

Whenever reasons are not given in the text, make sure you can give them yourself. In this proof in particular, give the reasons for the auxiliary lines and points in ① to ④.

(2) Given: $\overline{PX} \perp 1$

Prove: $\overline{PX}$ is the only line containing P and $\perp$ 1



Argument

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① Suppose there is another line containing P and $\perp$ 1. Let it intersect 1 at Y.

② Let Q be a point on $\overline{PY}$ such that QY = PY as shown.

③ Draw $\overline{QX}$

$$\text{①}\ PY \perp 1 \xrightarrow{\text{④}} \left.\begin{array}{r} \angle PYX \cong \angle QYX \\ \text{②}\ \overline{PY} \cong \overline{QY} \\ \text{⑤}\ \overline{YX} \cong \overline{YX} \end{array}\right\} \xrightarrow{\text{⑥}} \triangle PYX \cong \triangle QYX \xrightarrow{\text{⑦}}$$

$$\left.\begin{array}{r} \xrightarrow{\text{⑦}} \angle PXY \cong \angle QXY \\ \text{⑧}\ \angle PXY \text{ is a rt. } \angle \end{array}\right\} \rightarrow \cdots \xrightarrow{\text{⑨}} \left.\begin{array}{r} \angle QXY \text{ is a rt. } \angle \xrightarrow{\text{⑩}} \overline{QX} \perp 1 \text{ at } X \\ \text{⑪ But } \overline{PX} \perp 1 \text{ at } X \end{array}\right\} \xrightarrow{\text{⑫}}$$

$$\xrightarrow{\text{⑫}} \text{The supposition is false}$$

$\therefore$ PX is the only line containing P and $\perp$ 1.

Some comments are in order.

First, the figure looks rather ridiculous, doesn't it? This will generally be the case in an indirect proof because the figure will represent the conditions in the supposition which we hope to prove false. Try to rely on the logic rather than the figure.

Where the proof indicates $\rightarrow \cdots \xrightarrow{\text{⑨}}$, we are stating a rather obvious implication from ⑦ and ⑧, but which we have not stated as a formal theorem. This would have to be another agreement unless we wish to prove the theorem "If one of two congruent angles is a right angle, then the other one is a right angle". If you do use the indication shown in this proof to omit some "obvious" steps, you must make sure that you can supply both the statements and the reasons for them if you are challenged to do so.

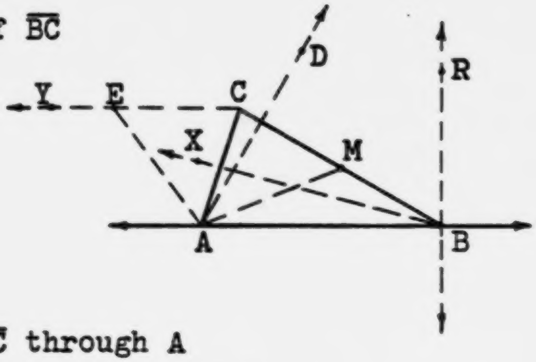
If you missed the "known fact" that is contradicted by ⑩ and ⑪, it is theorem 10-1. In a plane, at a given point on a line, there is exactly one line perpendicular to the given line.

We now have the two theorems justifying a unique perpendicular to a line; at a point on the line (thm 10-1), and through a point not on a line (thm 11-4). But notice that thm 10-1 is restricted to a plane since, in space, there would be an infinite number of lines perpendicular to a given line at a point on the line.

You may now add theorem 11-4 to the reasons listed in this section justifying auxiliary lines and points.

EXERCISE SET 11.5

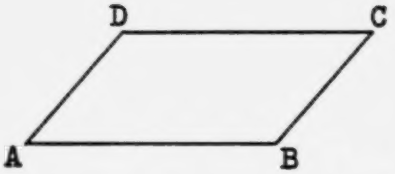
Give the reason that justifies the existence and uniqueness of each of the following auxiliary points, segments, rays, and lines.



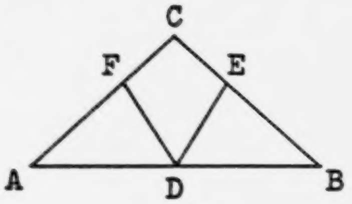
1. Let M be the midpoint of $\overline{BC}$
2. Draw $\overline{AM}$
3. Draw $\overleftrightarrow{BR}$ such that $\overleftrightarrow{BR} \perp \overleftrightarrow{AB}$ at B
4. Draw $\overrightarrow{CY}$ such that $m\angle ACY = m\angle CAB$
5. Draw $\overrightarrow{BX}$ such that $\overrightarrow{BX}$ bisects $\angle ABC$
6. Draw $\overrightarrow{AD}$ such that $\overrightarrow{AD} \perp \overline{BC}$ through A
7. Let E be a point on $\overrightarrow{CY}$ such that $CE = CA$
8. Draw $\overline{AE}$

Prove each of the following by introducing the appropriate auxiliary segment.

9. Given: $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$
   Prove: $\angle A \cong \angle C$



10. Given: $\overline{DF} \cong \overline{DE}$, $\overline{CF} \cong \overline{CE}$, $\angle A \cong \angle B$
    Prove: $\overline{AD} \cong \overline{BD}$



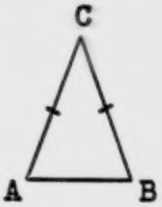
Theorems About Isosceles Triangles

The isosceles triangle is one of the more common geometric figures that we see around us. A gabled roof, a bridge truss, a steeple, a configuration of bowling pins are a few examples of representations of isosceles triangles. Because of its symmetry, it has important special properties which are generally acceptable by intuition, but which require proof and this gives us a further chance to examine proofs.

The isosceles triangle has already been defined as a triangle with at least two congruent sides.

If $\overline{AC} \cong \overline{BC}$, the terminology is as follows.

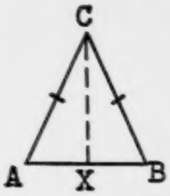
$\overline{AC}$ and $\overline{BC}$ are the legs.
$\overline{AB}$ is the base.
C is the vertex.
$\angle C$ is the vertex angle.
$\angle A$ and $\angle B$ are the base angles.



Theorem 11-5: If two sides of a triangle are congruent, then the angles opposite them are congruent.
(Base angles of an isosceles triangle are congruent.)

Given: $\overline{AC} \cong \overline{BC}$

Prove: $\angle B \cong \angle A$



Argument

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① Let X be the midpoint of $\overline{AB}$.
② Draw $\overline{CX}$

$$\left.\begin{array}{l} \textcircled{1}\ \overline{AX} \cong \overline{BX} \\ \textcircled{3}\ \overline{AC} \cong \overline{BC} \\ \textcircled{4}\ \overline{CX} \cong \overline{CX} \end{array}\right\} \xrightarrow{\textcircled{5}} \triangle ACX \cong \triangle BCX \xrightarrow{\textcircled{6}} \angle A \cong \angle B$$

$\therefore \overline{AC} \cong \overline{BC} \longrightarrow \angle B \cong \angle A$

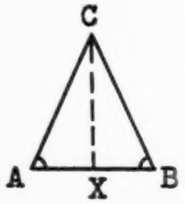
A corollary is a theorem which is proved directly from a major theorem and the major theorem is the primary reason used in the argument for the corollary.

Corollary 11-5a: Every equilateral triangle is equiangular.

The converse of theorem 11-5 is also very useful, but don't confuse the two with one another.

Theorem 11-6: If two angles of a triangle are congruent, then the sides opposite them are congruent.

Given: $\angle A \cong \angle B$
Prove: $\overline{BC} \cong \overline{AC}$



Argument

① Let $\overline{CX}$ be the angle bisector of $\angle ACB$ where X is the intersection with $\overline{AB}$.

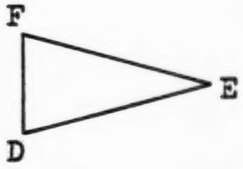
$$\left.\begin{array}{l} \textcircled{1}\ \angle ACX \cong \angle BCX \\ \textcircled{2}\ \angle A \cong \angle B \\ \textcircled{3}\ \overline{CX} \cong \overline{CX} \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle ACX \cong \triangle BCX \xrightarrow{\textcircled{5}} \overline{AC} \cong \overline{BC}$$

$$\therefore \angle A \cong \angle B \longrightarrow \overline{BC} \cong \overline{AC}$$

Corollary 11-6a: Every equiangular triangle is equilateral.

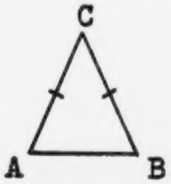
EXERCISE SET 11.6

$\triangle DEF$ is isosceles with $\overline{DE} \cong \overline{FE}$.



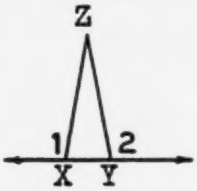
1. Name the base.
2. Name the vertex.
3. Name the base angles.
4. Name the legs.
5. Name the vertex angle.
6. Prove theorem 11-5 by introducing the angle bisector of the vertex angle as an auxiliary segment.
7. Prove corollary 11-5a.
8. Prove theorem 11-6 by introducing the perpendicular from the vertex to the base as an auxiliary segment.
9. Prove corollary 11-6a.

10. If $\overline{AC} \cong \overline{BC}$, then it can be proved that $\triangle ABC \cong \triangle BAC$. Use this fact to prove theorem 11-5.

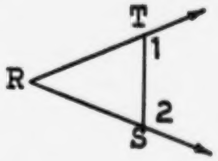


The purpose of proof exercises in any section is to give you the opportunity to use the knowledge gained in that particular section together with your accumulated knowledge of previous sections. Keep this in mind throughout the course.

11. Given: $\overline{XZ} \cong \overline{YZ}$
Prove: $\angle 1 \cong \angle 2$



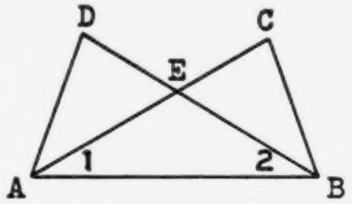
12. Given: $\angle 1 \cong \angle 2$
Prove: $\overline{RS} \cong \overline{RT}$



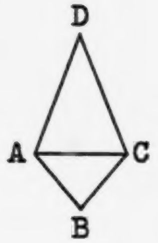
13. Write the statements below in proper flow proof form.

Given: $\overline{DE} \cong \overline{CE}$, $\angle 1 \cong \angle 2$
Prove: $\overline{AD} \cong \overline{BC}$

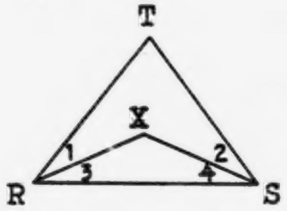
a) $\overline{DE} \cong \overline{CE}$
b) $\underline{\angle 1} \cong \underline{\angle 2}$
c) $\overline{AD} \cong \overline{BC}$
d) $\angle DEA \cong \angle CEB$
e) $\underline{\triangle ADE} \cong \triangle BCE$
f) $\overline{AE} \cong \overline{BE}$



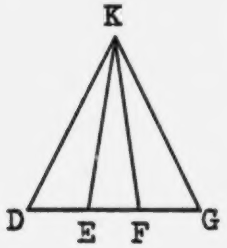
14. Given: $\overline{DA} \cong \overline{DC}$, $\angle DAB \cong \angle DCB$
Prove: $\overline{AB} \cong \overline{CB}$



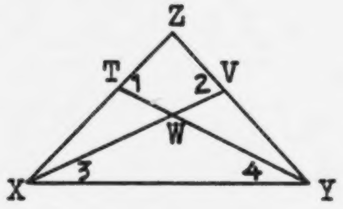
15. Given: $\overline{RT} \cong \overline{ST}$, $\angle 1 \cong \angle 2$
Prove: $\overline{RX} \cong \overline{SX}$



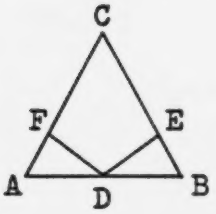
16. Given: $\overline{KD} \cong \overline{KG}$
$\overline{DE} \cong \overline{GF}$
Prove: $\overline{KE} \cong \overline{KF}$



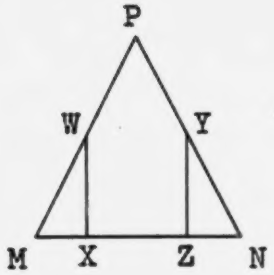
17. Given: $\overline{ZT} \cong \overline{ZV}$
$\angle 1 \cong \angle 2$
Prove: $\angle 3 \cong \angle 4$



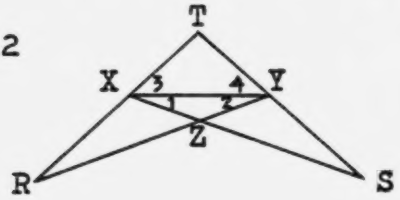
18. Given: $\overline{AD} \cong \overline{BD}$, $\overline{AF} \cong \overline{BE}$,
$\overline{DF} \cong \overline{DE}$
Prove: $\overline{CF} \cong \overline{CE}$



19. Given: W and Y are midpoints
of $\overline{MP}$ and $\overline{NP}$
$\overline{WX} \perp \overline{MN}$, $\overline{YZ} \perp \overline{MN}$, $\angle M \cong \angle N$
Prove: $\overline{WX} \cong \overline{YZ}$



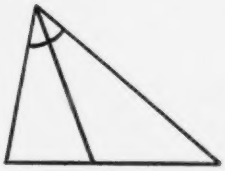
20. Given: $\overline{TX} \cong \overline{TY}$, $\angle 1 \cong \angle 2$
Prove: $\overline{RZ} \cong \overline{SZ}$



21. Construct an isosceles triangle. Label it $\triangle ABC$ with the vertex at C. Construct the perpendicular from the vertex to the base. Label the intersection with the base D. Construct the midpoints of $\overline{AC}$ and $\overline{BC}$. Call them E and F respectively. Draw $\overline{DE}$ and $\overline{DF}$. Using what you have constructed as the "Given", prove $\triangle AED \cong \triangle BFD$.

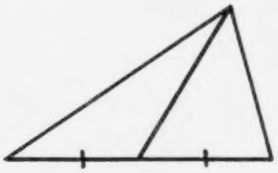
Special Segments and Lines Related to Triangles

We have already defined an angle bisector. Naturally every triangle has three angle bisectors, one at each vertex. We will show an angle bisector in a triangle as a segment from a vertex to the point of intersection with the opposite side.

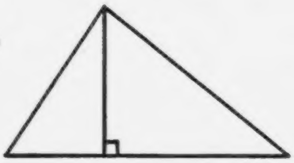


A median of a triangle is a segment from a vertex to the midpoint of the opposite side.

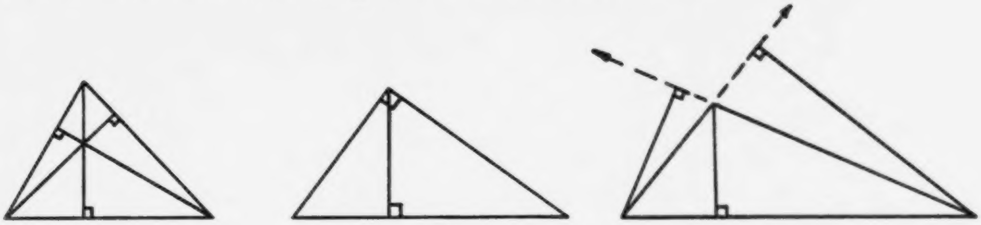
Of course, every triangle also has three medians, one at each vertex.



An altitude of a triangle is a segment from a vertex perpendicular to the line containing the opposite side.



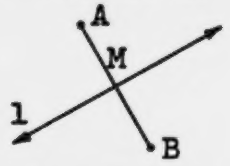
There are three cases involving altitudes of triangles that require attention. Notice the relationships of the altitudes to the triangle for the acute, right, and obtuse triangles. The case of the obtuse triangle is the reason for the wording in the definition.



acute Δ   right Δ   obtuse Δ

Once again, every triangle has three altitudes. What can you say about two of the altitudes of a right triangle?

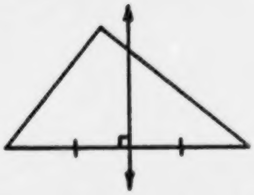
A line is the perpendicular bisector of a segment if and only if it is perpendicular to the segment at its midpoint.



$$l \text{ is } \perp \text{ bis. of } \overline{AB} \longleftrightarrow \begin{cases} l \perp \overline{AB} \text{ at } M \\ AM = MB \end{cases}$$

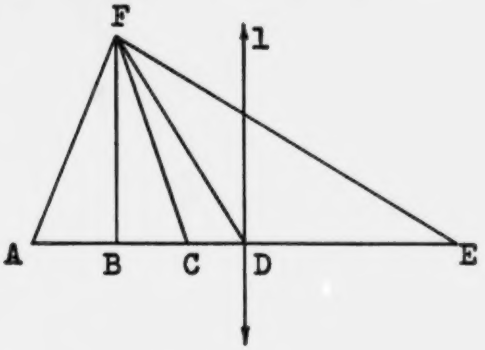
As with the definition of a bisector of a segment, a perpendicular bisector of a segment may be a ray or a segment if the line containing it is a perpendicular bisector.

When we consider the perpendicular bisectors of the sides of a triangle, we do not relate to a vertex as we do for the angle bisectors, medians, and altitudes.



All four of these lines or segments are very important in connection with special properties of triangles.

In the general scalene triangle shown below we have illustrated one of each to emphasize the fact that in general they are distinct from one another.



$\overline{FB} \perp \overline{AE} \longrightarrow \overline{FB}$ is an altitude

$\angle AFC \cong \angle EFC \longrightarrow \overline{FC}$ is an angle bisector

$AD = ED \longrightarrow \overline{FD}$ is a median

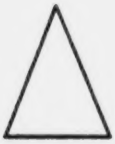
$\left.\begin{matrix} AD = ED \\ l \perp \overline{AE} \end{matrix}\right\} \longrightarrow$ l is a perpendicular bisector

Setting Up a Proof of a Theorem

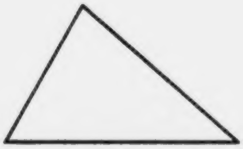
Most of the proof exercises you have had this far have provided you with the given (hypothesis), prove (conclusion), and a figure. In the formal demonstration of a theorem, you are required to draw and label the figure, and identify the given and prove before beginning the argument. This is called "setting up a proof".

The representation of the figure is one of the most critical stages in the entire proof. You must draw or construct a figure representing all the conditions of the *hypothesis* and only the conditions of the *hypothesis*. The conditions of the conclusion should appear to be present, but *not* because you draw or construct a figure that way.

One of the most common faults of students beginning proof is to draw a figure which "appears" to have more special properties than the hypothesis contains. For example, if a theorem is about a triangle, a common tendency is to draw an apparent isosceles triangle as shown on the left. But if the hypothesis does not specify an isosceles triangle, it should always be drawn as scalene as on the right.



incorrect representation
of a general triangle



correct representation
of a general triangle

The left figure would not actually be wrong, but the danger would be in assuming a property or relation that is not valid. The history of classroom geometry is full of instances of incorrect proofs because of false assumptions that were made. We call them "hidden assumptions" because the prover is generally not aware of the fact that they are being made.

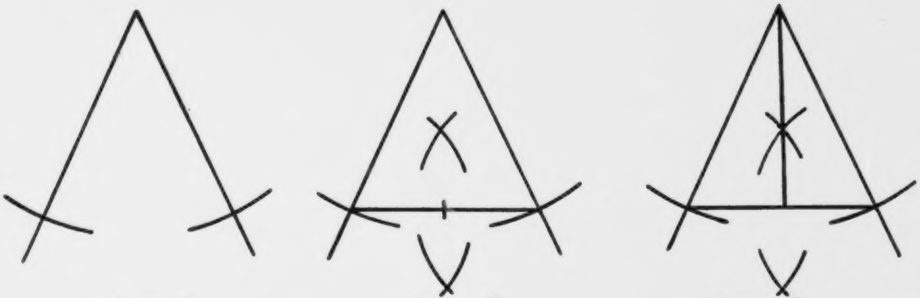
This section will concern itself primarily with theorems about the isosceles triangle, so we will have a special case to begin with. Consider the following theorem.

The median to the base of an isosceles triangle is perpendicular to the base.

Remember that the best way to identify the hypothesis and conclusion of a conditional statement is to restate it in the "If p, then Q" form.

If a triangle is isosceles and the median is drawn to the base, then it is perpendicular to the base.

In the most rigorous sense, every figure should be constructed according to the conditions of the hypothesis as shown here.

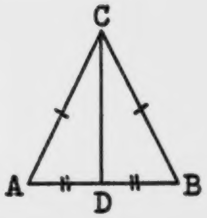


step 1 step 2 step 3

This may be awkward to do in every case but at times you may find it quite helpful. It emphasizes the fact that the given is that which was constructed. We now label our figure conveniently and identify the given and the prove specifically from the figure.

Given: $\overline{CA} \cong \overline{CB}$, $\overline{AD} \cong \overline{BD}$

Prove: $\overline{CD} \perp \overline{AB}$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ \overline{CA} \cong \overline{CB} \\ \textcircled{2}\ \overline{AD} \cong \overline{BD} \\ \textcircled{3}\ \overline{CD} \cong \overline{CD} \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle CAD \cong \triangle CBD \xrightarrow{\textcircled{5}} \angle CDA \cong \angle CDB \xrightarrow{\textcircled{6}} \overline{CD} \perp \overline{AB}$$

The theorem could have been stated as follows.

The median to the base of an isosceles triangle is an altitude.

This is just one of a large number of theorems about isosceles triangles which may be taken from the following very general statement.

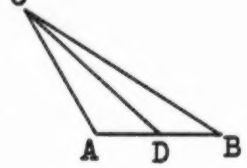
If a segment or line related to an isosceles triangle is any one of these four; the bisector of the vertex angle, the median to the base, the altitude to the base, or the perpendicular bisector of the base, then it is each of the other three.

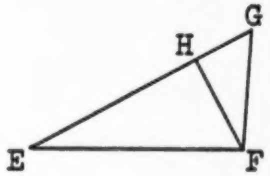
Of course, the perpendicular bisector of the base would be said to contain each of the other three. The exercises will give you a chance to become familiar with these theorems and will also give you practice in presenting the formal demonstration of a theorem.

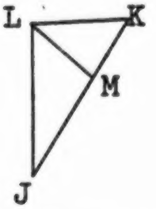
EXERCISE SET 11.7

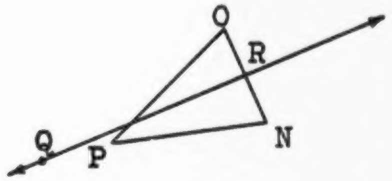
Using the figures shown, complete the following implications by naming the line or segment that is not part of the basic triangle.

1. $AD = BD \longrightarrow$
2. $\overline{FH} \perp \overline{GE} \longrightarrow$
3. $\angle JLM \cong \angle KLM \longrightarrow$
4. $\left.\begin{array}{l} OR = NR \\ \overline{RQ} \perp \overline{ON} \end{array}\right\} \longrightarrow$









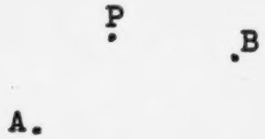
Give a formal demonstration of each of the following.

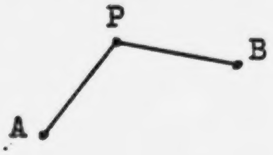
5. The bisector of the vertex angle of an isosceles triangle bisects the base.
6. The altitude to the base of an isosceles triangle bisects the vertex angle.
7. The median to the base of an isosceles triangle bisects the vertex angle.
8. The altitude to the base of an isosceles triangle bisects the base.
9. The bisector of the vertex angle of an isosceles triangle is perpendicular to the base.
10. The altitudes to the legs of an isosceles triangle are congruent.
11. The medians to the legs of an isosceles triangle are congruent.

Perpendicular Bisector Theorems

Two theorems related to perpendicular bisectors are worthy of special attention. We first introduce the word equidistant as a short way of saying equally distant.

We say that point P is equidistant from A and B if PA = PB.



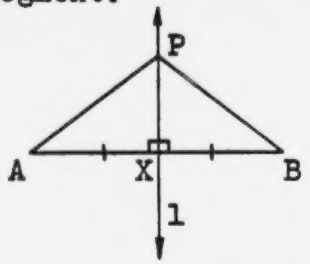


Notice that P does not have to be between A and B in order to be equidistant from A and B.

Theorem 11-7: If a point is on the perpendicular bisector of a segment, then it is equidistant from the endpoints of the segment.

Given: 1 is ⊥ bis. of $\overline{AB}$
P is any point on 1

Prove: PA = PB



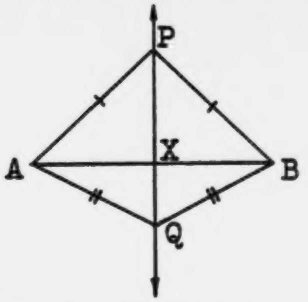
Argument

$$\begin{tikzcd}
\text{①}\ 1 \text{ is} \perp \text{bis. of } \overline{AB} \arrow[r, "\text{②}"] \arrow[dr, "\text{③}"'] & \angle PXA \cong \angle PXB \\
& \overline{XA} \cong \overline{XB} \arrow[r, "\text{⑤}"] & \triangle PXA \cong \triangle PXB \arrow[r, "\text{⑥}"] & PA = PB \\
& \text{④}\ \overline{PX} \cong \overline{PX}
\end{tikzcd}$$

$\therefore$ P is any point on ⊥ bis. of $\overline{AB} \longrightarrow PA = PB$

Notice that P is merely a representative of the infinite set of points on 1 and this argument is valid for all of them. If P is on $\overline{AB}$ we have a special case, but the conclusion follows directly since 1 bisects $\overline{AB}$.

Theorem 11-8: In a plane, if two points are each equidistant from the endpoints of a segment, the line containing them is the perpendicular bisector of the segment.



Given: $PA = PB$, $QA = QB$

Prove: $\overleftrightarrow{PQ}$ is the $\perp$ bis. of $\overline{AB}$

Argument

$$\left.\begin{array}{l} \textcircled{1}\ \overline{PA}\cong\overline{PB} \\ \textcircled{2}\ \overline{QA}\cong\overline{QB} \\ \textcircled{3}\ \overline{PQ}\cong\overline{PQ} \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle PAQ\cong\triangle PBQ \xrightarrow{\textcircled{5}} \left.\begin{array}{r} \angle APQ\cong\angle BPQ \\ \textcircled{1}\ \overline{PA}\cong\overline{PB} \\ \textcircled{6}\ \overline{PX}\cong\overline{PX} \end{array}\right\} \xrightarrow{\textcircled{7}} \triangle PAX\cong\triangle PBX \begin{array}{l} \xrightarrow{\textcircled{8}} \\ \xrightarrow{\textcircled{9}} \end{array}$$

$$\xrightarrow{\textcircled{8}} \angle PXA\cong\angle PXB \xrightarrow{\textcircled{10}} \left.\begin{array}{r} \overline{PX}\perp\overline{AB} \\ \xrightarrow{\textcircled{9}} \overline{XA}\cong\overline{XB} \end{array}\right\} \xrightarrow{\textcircled{11}} \overleftrightarrow{PQ} \text{ is } \perp \text{ bis. of } \overline{AB}$$

$$\therefore \left.\begin{array}{l} PA = PB \\ QA = QB \end{array}\right\} \longrightarrow \overleftrightarrow{PQ} \text{ is } \perp \text{ bis. of } \overline{AB}$$

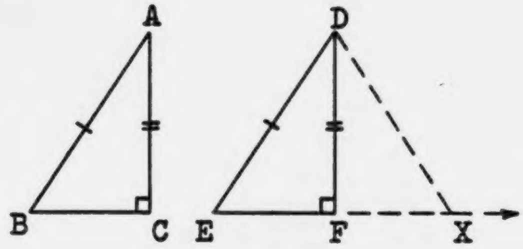
Another Congruence Theorem

It was stated earlier that a SSA correspondence between two triangles does not give us a congruence relation. But if the angle involved in that correspondence is a right angle, we do have a special case which leads to the following theorem.

Theorem 11-9: If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the triangles are congruent. (HL rt.Δ theorem)

Given: $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$
∠ACB is a rt.∠
∠DFE is a rt.∠

Prove: $\triangle ABC \cong \triangle DEF$



Argument

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① Extend $\overline{EF}$ beyond F.
② Let X be a point as shown such that FX = CB.
③ Draw $\overline{DX}$

$$\text{④ } \angle DFE \text{ is a rt.}\angle \rightarrow \cdots \xrightarrow{⑤} \left.\begin{array}{l} \angle DFX \text{ is a rt.}\angle \\ \text{⑥ } \angle ACB \text{ is a rt.}\angle \end{array}\right\} \xrightarrow{⑦} \left.\begin{array}{l} \angle DFX \cong \angle ACB \\ \text{②\ } \overline{FX} \cong \overline{CB} \\ \text{⑧\ } \overline{DF} \cong \overline{AC} \end{array}\right\} \xrightarrow{⑨}$$

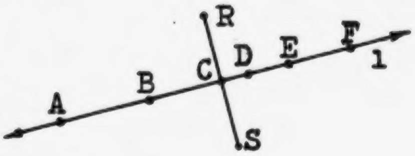
$$\xrightarrow{⑨} \triangle DXF \cong \triangle ABC \xrightarrow{⑩} \left.\begin{array}{l} \overline{DX} \cong \overline{AB} \\ \text{⑪ } \overline{DE} \cong \overline{AB} \end{array}\right\} \xrightarrow{⑫} \overline{DX} \cong \overline{DE} \xrightarrow{⑬} \left.\begin{array}{l} \angle DEF \cong \angle DXF \\ \cdots \text{⑭ } \angle DFE \cong \angle DFX \\ \text{⑮ } \overline{DF} \cong \overline{DF} \end{array}\right\} \xrightarrow{⑯}$$

$$\left.\begin{array}{l} \xrightarrow{⑯} \triangle DEF \cong \triangle DXF \\ \text{⑨ } \triangle DXF \cong \triangle ABC \end{array}\right\} \xrightarrow{⑰} \triangle ABC \cong \triangle DEF$$

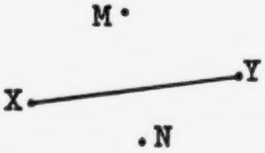
$$\therefore \left.\begin{array}{l} \overline{AB} \cong \overline{DE} \\ \overline{AC} \cong \overline{DF} \\ \angle C \text{ is a rt.}\angle \\ \angle F \text{ is a rt.}\angle \end{array}\right\} \rightarrow \triangle ABC \cong \triangle DEF$$

EXERCISE SET 11.8

1. If l is the perpendicular bisector of $\overline{RS}$, what can be said about all of the points A, B, C, D, E, and F?



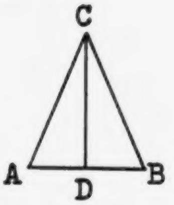
2. If M is equidistant from X and Y, and N is equidistant from X and Y, what conclusion can be stated about $\overleftrightarrow{MN}$?



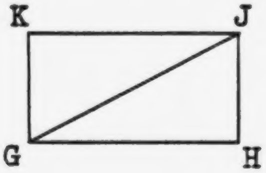
3. Write the reasons for all statements in the proof of theorem 11-9 except ①, ⑤, and ⑭.

4. Write all of the three part correspondences by which two triangles can be proved congruent.

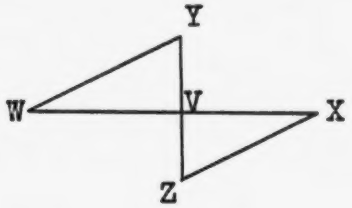
5. Given: $\overline{CD} \perp \overline{AB}$, $\overline{AC} = \overline{BC}$
Prove: $\angle ADC = \angle BDC$
(prove by using HL rt. Δ)



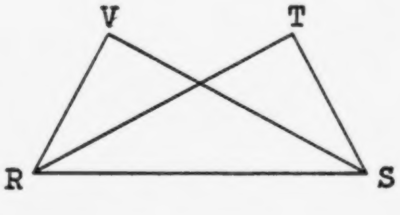
6. Given: $\overline{GK} \perp \overline{KJ}$, $\overline{GH} \perp \overline{HJ}$,
$\overline{GK} \cong \overline{JH}$
Prove: $\overline{GH} \cong \overline{JK}$



7. Given: $\angle WVZ \cong \angle XVZ$, $\overline{WY} \cong \overline{XZ}$,
$\overline{VW} \cong \overline{VX}$
Prove: $\angle Y \cong \angle Z$



8. Given: $\overline{RT} \cong \overline{SV}$, $\overline{RT} \perp \overline{ST}$,
$\overline{SV} \perp \overline{RV}$
Prove: $\overline{RV} \cong \overline{ST}$



CHAPTER 12 - GEOMETRIC INEQUALITIES
or I'M GREATER THAN YOU ARE

Inequality Theorems

There are some important theorems and consequences in geometry regarding inequalities as well as equalities. They are mostly acceptable on the basis of intuition but their proofs will help to emphasize their importance as well as give you an opportunity to look at proof in a different situation.

The main properties of inequality were stated earlier, but we will review them here for convenience.

If $a = b + c$ and c is positive, then $b < a$

If $a < b$, then $b > a$

If $a < b$ and $b < c$, then $a < c$

If $a < b$, then $a + c < b + c$

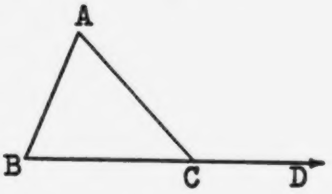
If $a < b$ and $c > 0$, then $ac < bc$

If $a < b$ and $c < 0$, then $ac > bc$

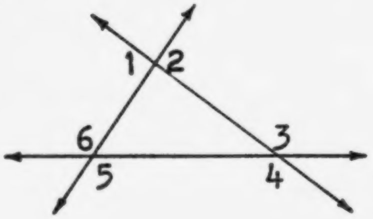
If a and b are real numbers, exactly one of the following is true, $a < b$, $a = b$, $a > b$. (Trichotomy)

If one side of a triangle is extended beyond a vertex, the new angle formed is called an exterior angle of the triangle. $\angle ACD$ is an *exterior angle* of $\triangle ABC$.

With respect to this particular exterior angle, $\angle ACB$ is the *adjacent interior angle*, and $\angle CAB$ and $\angle CBA$ are called *remote interior angles*.



Every triangle would have six exterior angles as shown. Any pair at the same vertex are congruent since they are vertical angles.

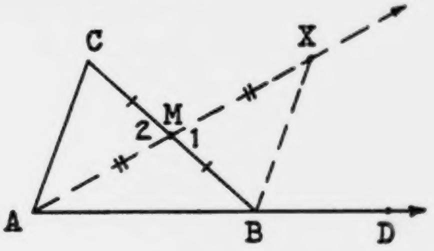


The following theorem has a great deal of significance for later developments.

Theorem 12-1: The measure of an exterior angle of a triangle is greater than the measure of each remote interior angle.

Given: $\triangle ABC$ with ext. $\angle CBD$

Prove: $m\angle CBD > m\angle C$
$m\angle CBD > m\angle A$

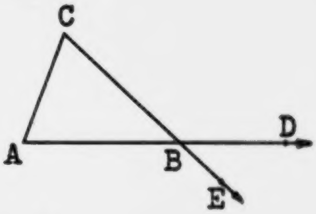


Argument

① Let M be the midpoint of $\overline{BC}$.
② Draw $\overline{AM}$
③ Let X be a point as shown such that $MX = MA$
④ Draw $\overline{BX}$

$$\left.\begin{array}{r}\left.\begin{array}{l}\textcircled{1}\ \overline{BM}\cong\overline{CM}\\ \textcircled{3}\ \overline{XM}\cong\overline{AM}\\ \textcircled{5}\ \angle 1\cong\angle 2\end{array}\right\}\xrightarrow{\textcircled{6}}\triangle BMX\cong\triangle CMA\xrightarrow{\textcircled{7}} m\angle MBX = m\angle MCA\\ \textcircled{8}\ m\angle MBD = m\angle MBX + m\angle XBD\xrightarrow{\textcircled{9}} m\angle MBD > m\angle MBX\end{array}\right\}\xrightarrow{\textcircled{10}} m\angle MBD > m\angle MCA$$

The other exterior angle formed by extending $\overline{CB}$ beyond B could be proved in a similar manner to have a measure greater than $m\angle A$. Since $\angle CBD$ and $\angle ABE$ are vertical angles, $m\angle CBD$ must also be greater than $m\angle A$.

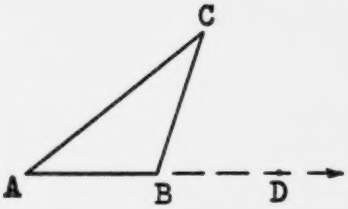


$\therefore \angle CBD$ is an ext. $\angle$ of $\triangle ABC \begin{array}{l}\nearrow m\angle CBD > m\angle C\\ \searrow m\angle CBD > m\angle A\end{array}$

Corollary 12-1a: If one angle of a triangle is right or obtuse, the other two angles are acute.

Given: $m\angle ABC \geq 90$

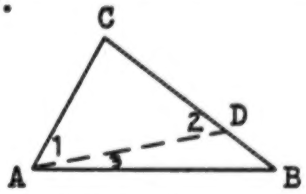
Prove: $m\angle C < 90$
$m\angle A < 90$



Informal argument

Since $\angle ABC$ and $\angle CBD$ are a linear pair, the sum of their measures is 180. $m\angle ABC \geq 90 \rightarrow \ldots \rightarrow m\angle CBD \leq 90$. But by the previous theorem, $m\angle CBD > m\angle A$ and $m\angle CBD > m\angle C$. $\therefore m\angle A < 90$ and $m\angle C < 90$.

Theorem 12-2: If two sides of a triangle are not congruent, the angles opposite those sides are not congruent and the angle opposite the longer side is the larger.



Given: $BC > AC$

Prove: $m\angle A > m\angle B$

Argument

---

① Let D be a point on $\overline{BC}$ such that $CD = CA$.
② Draw $\overline{AD}$

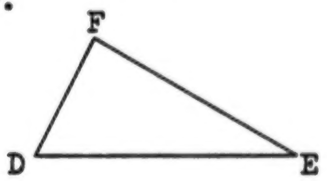
$$\left.\begin{array}{l} \text{①}\ CD = CA \xrightarrow{\text{③}} \left.\begin{array}{l} m\angle 1 = m\angle 2 \\ \text{④}\ m\angle 2 > m\angle B \end{array}\right\} \xrightarrow{\text{⑤}} m\angle 1 > m\angle B \\ \text{⑥}\ m\angle CAB = m\angle 1 + m\angle 3 \xrightarrow{\text{⑦}} m\angle CAB > m\angle 1 \end{array}\right\} \xrightarrow{\text{⑧}} m\angle CAB > m\angle B$$

$\therefore BC > AC \longrightarrow m\angle A > m\angle B$

Statement ④ is a key link in the proof so make sure you can see the reason for it along with the others.

The converse of theorem 12-2 would seem to require a similar proof, but we find it necessary to use an indirect proof and make application of the Law of the Trichotomy.

Theorem 12-3: If two angles of a triangle are not congruent, the sides opposite those angles are not congruent and the side opposite the larger angle is the longer.



Given: $m\angle D > m\angle E$

Prove: $EF > DF$

Argument by indirect proof

---

① Suppose $EF \ngtr DF$
Then either ② $EF = DF$ or ③ $EF < DF$

$$\left.\begin{array}{l} \text{②}\ EF = DF \xrightarrow{\text{④}} \left.\begin{array}{l} m\angle D = m\angle E \\ \text{⑤ But } m\angle D > m\angle E \end{array}\right\} \xrightarrow{\text{⑥}} EF = DF \text{ is false} \\ \text{③}\ EF < DF \xrightarrow{\text{⑦}} \left.\begin{array}{l} m\angle D < m\angle E \\ \text{⑤ But } m\angle D > m\angle E \end{array}\right\} \xrightarrow{\text{⑧}} EF < DF \text{ is false} \end{array}\right\} \xrightarrow{\text{⑨}}$$

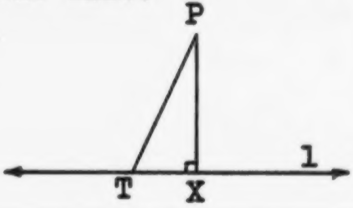
$\xrightarrow{\text{⑨}} EF \ngtr DF$ is false and $EF > DF$ is true

$\therefore m\angle D > m\angle E \longrightarrow EF > DF$

Corollary 12-3a: If a point is not on a line, the shortest distance from the point to the line is the length of the perpendicular segment from the point to the line.

Given: P not on l, $\overline{PX} \perp l$
$\overline{PT}$ is any other segment $\not\perp l$

Prove: $PX < PT$



The proof is left as an exercise.

Definition: The distance from a point to a line is the length of the perpendicular segment from the point to the line.

This is the first definition for distance given other than the distance between two points. In general, any later definitions for distance between any two figures will follow the same pattern of shortest distance. This should seem reasonable.

Corollary 12-3b: The hypotenuse of a right triangle is the longest side of the triangle.

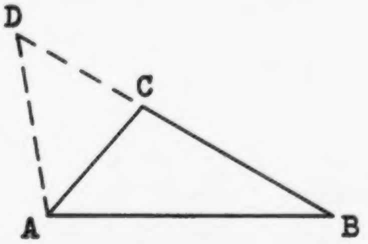
You have probably heard the saying "the shortest distance between two points is a straight line". We modify it slightly and say "the shortest distance between two points is the length of the segment joining them". That is, we know that the shortest route from A to B is by way of $\overline{AB}$ rather than any other route we might take.

This is the principle involved in the next theorem which is generally called the Triangle Inequality.

Theorem 12-4: The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Given: $\triangle ABC$

Prove: $AC + CB > AB$



Argument

---

① Extend $\overline{BC}$ beyond C.
② Let D be a point as shown such that $CD = CA$.
③ Draw $\overline{DA}$

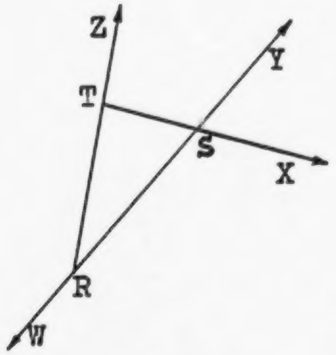
$$\textcircled{4}\; \left.\begin{array}{r} m\angle DAB = m\angle DAC + m\angle CAB \xrightarrow{\textcircled{5}} m\angle DAB > m\angle DAC \\ \textcircled{2}\; CD = CA \xrightarrow{\textcircled{6}} m\angle D = m\angle DAC \end{array}\right\} \xrightarrow{\textcircled{7}} m\angle DAB > m\angle D \xrightarrow{\textcircled{8}}$$

$$\left.\begin{array}{r} \xrightarrow{\textcircled{8}} DB > AB \\ \textcircled{9}\; DB = DC + CB \end{array}\right\} \xrightarrow{\textcircled{10}} \left.\begin{array}{r} DC + CB > AB \\ \textcircled{2}\; DC = AC \end{array}\right\} \xrightarrow{\textcircled{11}} AC + CB > AB$$

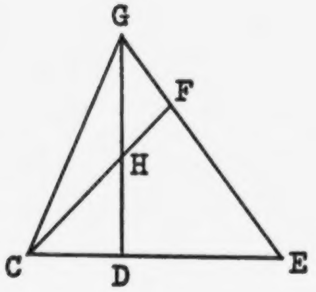
$\therefore$ In $\triangle ABC$, $AC + CB > AB$

Notice how statement ⑧ is a key statement in this proof.

EXERCISE SET 12.1



1. Name an exterior angle of $\triangle RST$ with vertex R.
2. Name two exterior angles of $\triangle RST$ with vertex S.
3. With respect to the exterior angle, $\angle STZ$, name the adjacent interior angle. Name the two remote interior angles.
4. Is $\angle XSY$ an exterior angle of $\triangle RST$?
5. With respect to $\triangle RSX$, name the adjacent interior angle. Name the two remote interior angles.
6. Is it ever possible that an exterior angle of a triangle and its adjacent interior angle would not be a linear pair?



7. Name an exterior angle of $\triangle CFG$.

8. Is $\angle FCG$ an exterior angle of $\triangle FCE$?

9. With respect to exterior angle, $\angle CDG$, of $\triangle GDE$, name the two remote interior angles.

10. Name an exterior angle of $\triangle GHC$.

11. Name two triangles for which $\angle DHF$ is an exterior angle.

12. $m\angle CHD = 40 \longrightarrow m\angle HCG$ ___?___

13. $m\angle GFH = 100 \longrightarrow m\angle GHC$ ___?___

14. $m\angle GFC = 100 \longrightarrow m\angle FEC$ ___?___

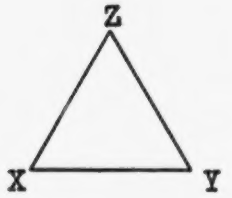
15. $m\angle GDE = 90 \longrightarrow m\angle GCD$ ___?___

16. $\angle GDC$ is a rt. $\angle \longrightarrow \angle GCD$ is ___?___

17. $\angle GFH$ is an obtuse $\angle \longrightarrow \angle GHF$ is ___?___

18. Write all the reasons in the proof of theorem 12-1.

19. Write all the reasons in the proof of theorem 12-2.



20. $XY > XZ \longrightarrow$?

21. $m\angle Y < m\angle X \longrightarrow$?

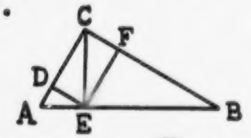
22. $\left.\begin{matrix} m\angle Z = 61 \\ m\angle X = 60 \end{matrix}\right\} \longrightarrow$?

23. $\left.\begin{matrix} YZ = 15 \\ XZ = 14 \end{matrix}\right\} \longrightarrow$?

24. Prove corollary 12-3a.

25. Prove corollary 12-3b.

In the figure, $\overline{CE} \perp \overline{AB}$, $\overline{ED} \perp \overline{CA}$, $\overline{EF} \perp \overline{BC}$, $\overline{BC} \perp \overline{CA}$. Name the distances between the following.



26. E and $\overline{AC}$

27. E and $\overline{BC}$

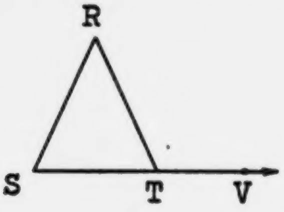
28. C and $\overline{AB}$

29. A and $\overline{BC}$

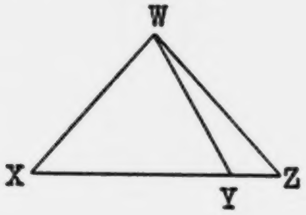
30. Write the reasons for statements ② through ⑪ in the proof of theorem 12-4.

31. If the lengths of two sides of a triangle are 5 and 8, then the length of the third side is less than ___?___ and more than ___?___.

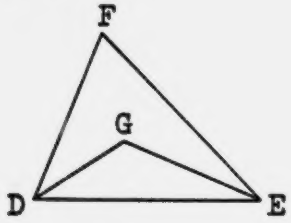
32. Given: $\overline{RS} \cong \overline{RT}$
Prove: $m\angle RTV > m\angle RTS$



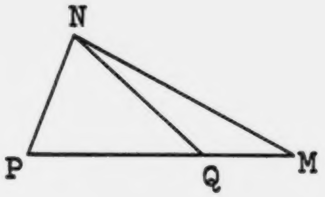
33. Given: $WX = WZ$
Prove: $WX > WY$



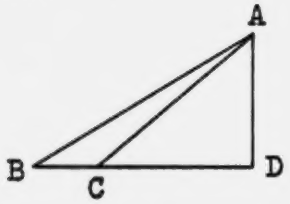
34. Given: $\overline{DG}$ bisects $\angle FDE$
$\overline{EG}$ bisects $\angle FED$
$EF > DF$
Prove: $EG > DG$



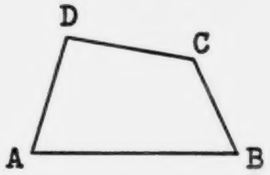
34. Given: $NQ = PQ$
Prove: $MP > MN$



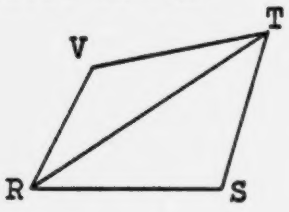
36. Given: $\overline{AD} \perp \overline{BD}$
Prove: $AB > AC$



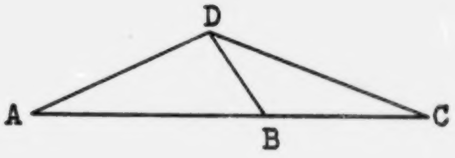
37. Given: quad. ABCD
Prove: $AD + DC + CB > AB$



38. Given: $m\angle VRS = m\angle VTS$
$VT > VR$
Prove: $ST < SR$



39. Given: $AB = DC$
Prove: $AD > BC$



40. Prove: The length of a median of a triangle is less than half the sum of the lengths of the sides.

CHAPTER 13 - PARALLEL LINES
or LET'S TAKE A PLANE

Definitions

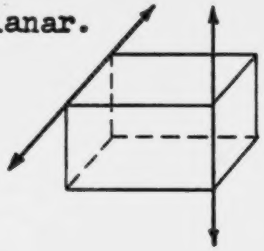
You've been familiar with the idea of parallel lines for quite some time. The top and bottom edges of this page are examples of two parallel lines in a physical sense. If a large number of people were asked to give their own example of parallel lines, the most common answer would probably be railroad tracks. They give us a good illustration of two lines that extend indefinitely and never meet, which brings us to our definition.

Parallel lines are lines which are coplanar and do not intersect.

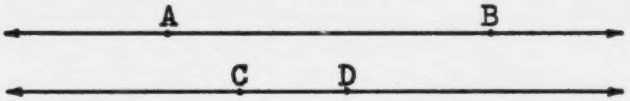
The coplanar condition is an important one. There are two possible relations for two coplanar lines, parallel and intersecting. Then there is a third relation for two lines in space. They may be noncoplanar, which gives us another definition.

Skew lines are lines which are noncoplanar.

This means that there does not exist any plane which can contain the two lines as in the figure shown.



We define rays and segments to be parallel if the lines containing them are parallel. The symbolism for parallel lines is $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ for the figure below. For this case we could also write $\overline{AB} \parallel \overrightarrow{CD}$, $\overrightarrow{AB} \parallel \overrightarrow{CD}$, $\overline{AB} \parallel \overrightarrow{CD}$, etc.

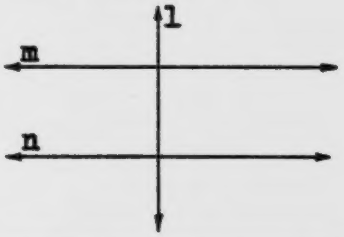


A thorough exploration of parallel lines leads to many new and interesting developments. The first part of the chapter will be concerned with two basic ideas; (1) under what conditions, besides the definition, are lines determined to be parallel, and (2) if lines are known to be parallel, what consequences can be deduced. Once again, some of these investigations can most appropriately be made by indirect proof.

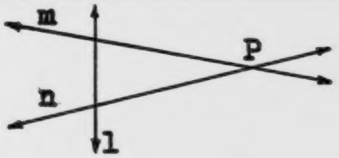
Theorem 13-1: In a plane, if two lines are both perpendicular to the same line, they are parallel.

Given: l, m, and n are coplanar
$m \perp l$, $n \perp l$

Prove: $m \parallel n$



Indirect argument



① Suppose $m \not\parallel n$

① $m \not\parallel n \xrightarrow{\text{②}}$ m and n intersect
Call point of intersection P

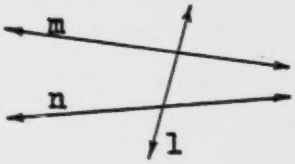
$$\left.\begin{array}{l} \text{③}\ m \perp l \text{ through } P \\ \text{④}\ n \perp l \text{ through } P \end{array}\right\} \xrightarrow{\text{⑤}} m \not\parallel n \text{ is false and } m \parallel n \text{ is true}$$

$\therefore$ In a plane, $\left.\begin{array}{l} m \perp l \\ n \perp l \end{array}\right\} \longrightarrow m \parallel n$

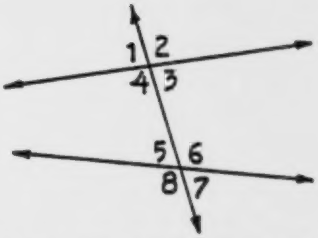
The reason for ⑤ is, of course, that ③ and ④ contradict the theorem that through a point outside a line, there is exactly one line perpendicular to that line.

Transversal Pairs of Angles

A transversal is a line that intersects two coplanar lines in two distinct points.



l is a transversal of m and n. Of course, this requires all three lines to be coplanar.



In every such case, there are obviously four angles at each point of intersection, or eight angles in all. We will call any pair of angles, one at each vertex, a transversal pair. You should recognize that there are always 16 transversal pairs of angles as in the figure shown here.

Since we are going to be particularly concerned with these pairs, we will give each pair its own special name as follows.

$\angle 1$ and $\angle 5$ are corresponding angles. Likewise $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, $\angle 4$ and $\angle 8$.

$\angle 3$ and $\angle 5$ are alternate interior angles. Likewise $\angle 4$ and $\angle 6$.

$\angle 1$ and $\angle 7$ are alternate exterior angles. Likewise $\angle 2$ and $\angle 8$.

$\angle 6$ and $\angle 1$ are alternate interior exterior angles. Likewise $\angle 5$ and $\angle 2$, $\angle 4$ and $\angle 7$, $\angle 3$ and $\angle 8$.

$\angle 3$ and $\angle 6$ are consecutive interior angles. Likewise $\angle 4$ and $\angle 5$. Here we take the word "consecutive" to mean "on the same side of the transversal". One might consider $\angle 2$ and $\angle 3$ to be consecutive, but remember, these are called a linear pair. We are talking here only about transversal pairs, that is, one at each vertex, so there should be no confusion.

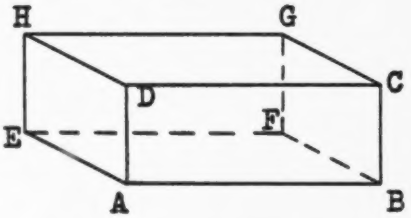
$\angle 1$ and $\angle 8$ are consecutive exterior angles. Likewise $\angle 2$ and $\angle 7$.

There --- we have named all 16 pairs. Make sure you are familiar with them.

EXERCISE SET 13.1

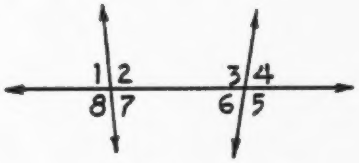
Using the appearance of the figure, tell whether each of the following are intersecting, parallel, or skew.

1. $\overline{AD}$ and $\overline{BC}$
2. $\overline{AE}$ and $\overline{FG}$
3. $\overline{GC}$ and $\overline{AB}$
4. $\overline{HE}$ and $\overline{GH}$
5. $\overline{DC}$ and $\overline{EF}$
6. $\overline{HD}$ and $\overline{AD}$

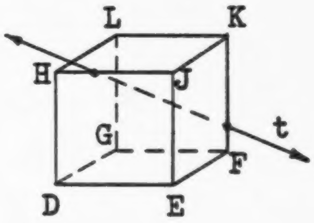


Give the appropriate name for each transversal pair.

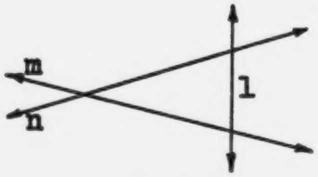
7. $\angle 1$ and $\angle 5$
8. $\angle 7$ and $\angle 3$
9. $\angle 5$ and $\angle 8$
10. $\angle 2$ and $\angle 3$
11. $\angle 3$ and $\angle 8$
12. $\angle 6$ and $\angle 8$



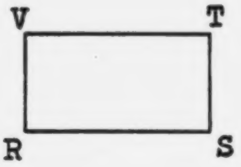
13. Is line t a transversal of $\overline{HJ}$ and $\overline{KF}$? Why or why not?



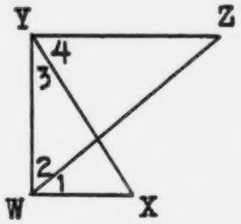
14. l, m, and n are coplanar lines.
Is l a transversal of m and n?
Is m a transversal of l and n?
Is n a transversal of l and m?



15. Given: ∠R is a rt. ∠
∠S is a rt. ∠
Prove: $\overline{VR} \parallel \overline{TS}$



16. Given: ∠1 compl. ∠2
∠3 compl. ∠4
Prove: $\overline{WX} \parallel \overline{YZ}$

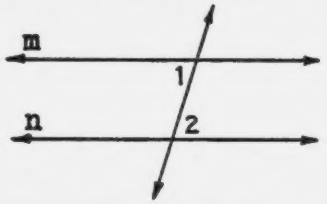


Proving Lines Parallel

Theorem 13-2: If two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.

Given: $\angle 1 \cong \angle 2$

Prove: $m \parallel n$



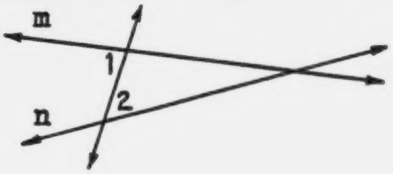
Indirect argument

---

① Suppose $m \nparallel n$

① $m \nparallel n \xrightarrow{②} m$ and n intersect

$\left.\begin{array}{l} \text{③ } m\angle 1 > m\angle 2 \\ \text{④ But } \angle 1 \cong \angle 2 \end{array}\right\} \xrightarrow{⑤} m \nparallel n$ is false and $m \parallel n$ is true



$\therefore \angle 1 \cong \angle 2 \longrightarrow m \parallel n$

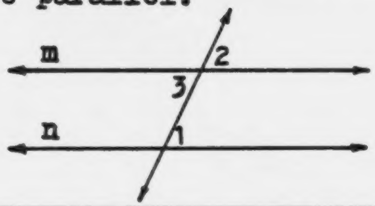
③ The measure of an exterior angle of a triangle is greater than the measure of each remote interior angle.
④ Given

The other pair of alternate interior angles would imply parallel lines in a similar manner.

Theorem 13-3: If two lines are cut by a transversal so that corresponding angles are congruent, then the lines are parallel.

Given: $\angle 1 \cong \angle 2$

Prove: $m \parallel n$



Argument

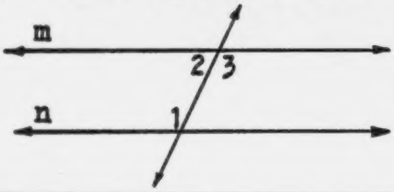
$$\left.\begin{array}{l} \textcircled{1}\ \angle 1 \cong \angle 2 \\ \textcircled{2}\ \angle 2 \cong \angle 3 \end{array}\right\} \xrightarrow{\textcircled{3}} \angle 1 \cong \angle 3 \xrightarrow{\textcircled{4}} m \parallel n$$

$$\therefore\ \angle 1 \cong \angle 2 \longrightarrow m \parallel n$$

Theorem 13-4: If two lines are cut by a transversal so that consecutive interior angles are supplementary, the lines are parallel.

Given: $\angle 1$ suppl. $\angle 2$

Prove: $m \parallel n$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ \angle 1 \text{ suppl. } \angle 2 \\ \textcircled{2}\ \angle 3 \text{ suppl. } \angle 2 \end{array}\right\} \xrightarrow{\textcircled{3}} \angle 1 \cong \angle 3 \xrightarrow{\textcircled{4}} m \parallel n$$

$$\therefore\ \angle 1 \text{ suppl. } \angle 2 \longrightarrow m \parallel n$$

In mathematics, we are constantly looking for generalizations that we can make based on a pattern we see in examples we've already done. Consider the following speculation.

If two lines are cut by a transversal so that any transversal pair of angles are either congruent or supplementary, then the lines are parallel.

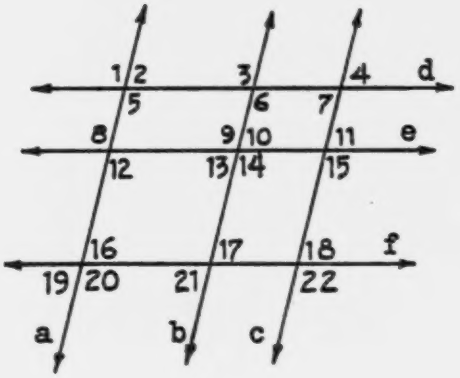
Do you think it is true?

EXERCISE SET 13.2

1. Prove: If two lines are cut by a transversal so that alternate exterior angles are congruent, the lines are parallel.
2. Prove: If two lines are cut by a transversal so that alternate interior exterior angles are supplementary, the lines are parallel.

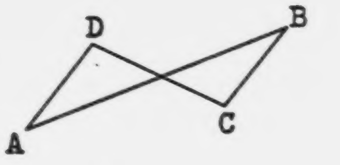
Complete each implication with a statement of parallel lines if possible.

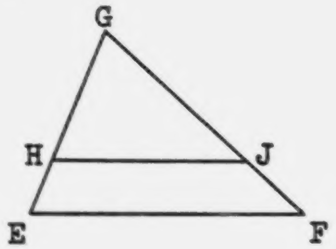
3. $\angle 3 \cong \angle 5 \longrightarrow$
4. $\angle 4 \cong \angle 11 \longrightarrow$
5. $\angle 12$ suppl. $\angle 16 \longrightarrow$
6. $\angle 8 \cong \angle 15 \longrightarrow$
7. $\angle 5 \cong \angle 9 \longrightarrow$
8. $\angle 14$ suppl. $\angle 18 \longrightarrow$
9. $\angle 4 \cong \angle 10 \longrightarrow$
10. $\angle 13$ suppl. $\angle 15 \longrightarrow$
11. $\angle 16 \cong \angle 18 \longrightarrow$
12. $\angle 7 \cong \angle 18 \longrightarrow$
13. $\angle 6$ suppl. $\angle 13 \longrightarrow$
14. $\angle 1 \cong \angle 22 \longrightarrow$



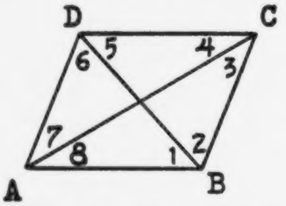
Remember that a figure involving transversal pairs in the exercises will not always look like the basic figure of theorem 13-2.

15. Consider $\overline{AB}$ as a transversal of $\overline{AD}$ and $\overline{BC}$. Name a pair of alternate interior angles.
16. Consider $\overline{CD}$ as a transversal of $\overline{AD}$ and $\overline{BC}$. Name a pair of alternate interior angles.
17. Consider $\overline{GE}$ as a transversal of $\overline{HJ}$ and $\overline{EF}$. Name a pair of corresponding angles.
18. Consider $\overline{GF}$ as a transversal of $\overline{HJ}$ and $\overline{EF}$. Name a pair of corresponding angles.





Complete each implication with a statement of parallel lines if possible.



19. $\angle 1 \cong \angle 5 \longrightarrow$

20. $\angle 2 \cong \angle 6 \longrightarrow$

21. $\angle 4 \cong \angle 1 \longrightarrow$

22. $\angle ADC$ suppl. $\angle DAB \longrightarrow$

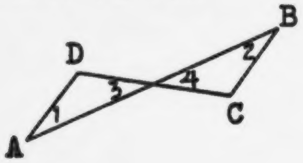
23. $\angle 7 \cong \angle 3 \longrightarrow$

24. $\angle 6 \cong \angle 7 \longrightarrow$

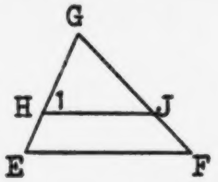
25. $\angle BAD$ suppl. $\angle ABC \longrightarrow$

26. $\angle 4 \cong \angle 8 \longrightarrow$

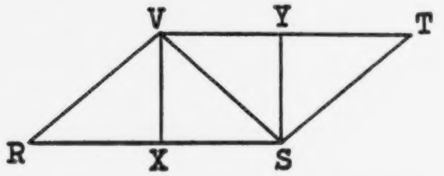
27. Given: $\angle 1 \cong \angle 3$, $\angle 2 \cong \angle 4$
Prove: $\overline{AD} \parallel \overline{CB}$



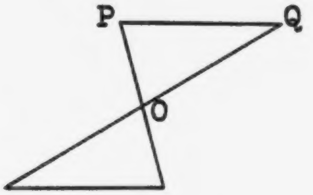
28. Given: $\overline{EF} \cong \overline{GF}$, $\angle G \cong \angle 1$
Prove: $\overline{HJ} \parallel \overline{EF}$



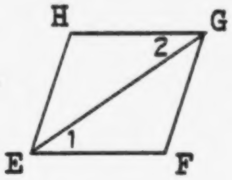
29. Given: $\angle RVS \cong \angle VST$
$\overline{VX}$ bisects $\angle RVS$
$\overline{SY}$ bisects $\angle VST$
Prove: $\overline{VX} \parallel \overline{YS}$



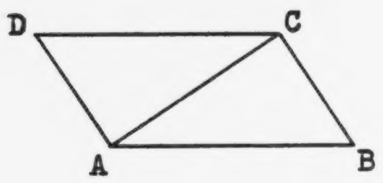
30. Given: $\overline{PO} \cong \overline{NO}$, $\overline{MO} \cong \overline{QO}$
Prove: $\overline{MN} \parallel \overline{PQ}$



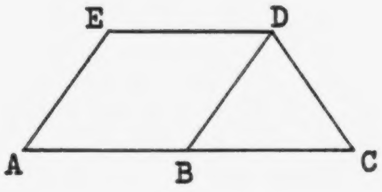
31. It is given that $\angle 1 \cong \angle 2$.
It appears that $\overline{EF} \parallel \overline{HG}$ and $\overline{EH} \parallel \overline{FG}$.
Complete the implication, $\angle 1 \cong \angle 2 \longrightarrow$
Draw another figure with $\angle 1 \cong \angle 2$ in which it does not appear that $\overline{EH} \parallel \overline{FG}$.



32. Given: $\overline{AD} \cong \overline{CB}$, $\overline{CD} \cong \overline{AB}$
Prove: $\overline{AD} \parallel \overline{BC}$, $\overline{DC} \parallel \overline{AB}$

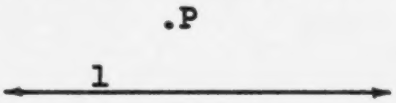


33. Given: $\overline{BD} \cong \overline{CD}$, $\angle C \cong \angle A$
Prove: $\overline{AE} \parallel \overline{BD}$



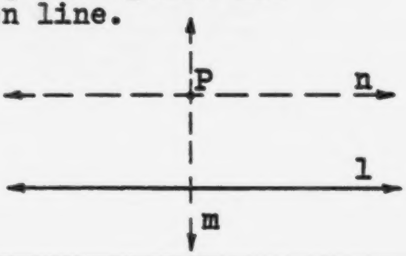
The Parallel Postulate

Use your intuition to answer the following question. If a point is not on a line, how many, if any, lines are there through the given point and parallel to the given line? That is, given P not on l, how many, if any, lines are there through P and parallel to l?



Though it may seem strange to you now, the consideration of this question has led to some of the most significant developments in the history of mathematics. It's a question of existence and uniqueness again, and we will first consider the question of existence which we can prove.

Theorem 13-5: If a point is not on a line, there is at least one line through the point and parallel to the given line.



Given: P not on l

Prove: There is at least one line through P and parallel to l

Argument

① Let m ⊥ l through P } ③ → n ∥ l
② Let n ⊥ l at P }

∴ There is at least one line, n, through P and parallel to l

① Through a point outside a line, there is exactly one line ⊥ to that line. (Thm 11-4)
② In a plane, at a given point on a line, there is exactly one line ⊥ to the given line. (Thm 10-1)
③ In a plane, if two lines are both ⊥ to the same line, they are ∥. (Thm 13-1)

Having established that there is at least one parallel, we would now consider uniqueness, "Is it the only one?". When Euclid considered this question, it seemed reasonable to him that the answer is yes. His fifth postulate, which is now famous for this reason, was equivalent to stating that through a point outside a line, there is exactly one line parallel to the given line. For centuries after Euclid, mathematicians did not doubt the truth of the statement, but some felt that it could possibly be proved as a theorem instead of accepted as a postulate. Beginning about 1733, attempts were made to prove the statement by indirect proof. The attempts actually failed in their intended purpose, but they succeeded in showing that a valid geometry can exist in which there are many parallels to a given line through a point not on the line. Any such geometry which contradicts Euclid's fifth postulate is called a Non-Euclidean geometry. They are perfectly acceptable systems and applications have been made in our space world.

However, we are going to stick with Euclid for the remainder of this course and accept the following statement without proof.

Postulate 13: Through a point not on a line, there is not more than one line parallel to the given line. (The Parallel Postulate)

If we put this postulate together with theorem 13-5, we get the statement below which is frequently stated as a reason in proofs.

Through a point not on a line, there is exactly one line parallel to the given line.

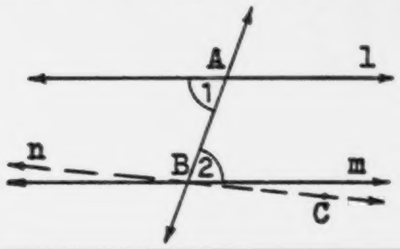
Consequences of Parallel Lines

We are now able to consider what theorems can be deduced when lines are known to be parallel. These are the converses of the earlier theorems. We begin with the converse of theorem 13-2.

Theorem 13-6: If two parallel lines are cut by a transversal, then alternate interior angles are congruent.

Given: $l \parallel m$

Prove: $\angle 1 \cong \angle 2$



Indirect argument

① Suppose $\angle 1 \ncong \angle 2$

② Let n be a line through B, distinct from m, such that $\angle ABC \cong \angle 1$.

$$\text{②}\ \left.\begin{array}{l} \angle ABC \cong \angle 1 \xrightarrow{\text{③}} n \parallel l \text{ through } B \\ \text{④}\ m \parallel l \text{ through } B \end{array}\right\} \xrightarrow{\text{⑤}} \begin{array}{l} \angle 1 \ncong \angle 2 \text{ is false} \\ \text{and } \angle 1 \cong \angle 2 \text{ is true} \end{array}$$

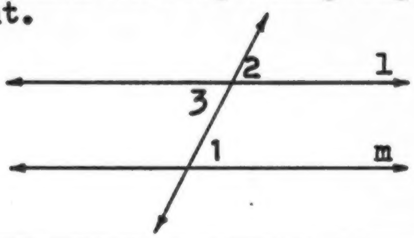
$\therefore l \parallel m \longrightarrow \angle 1 \cong \angle 2$

Reason ② is our first use of the Angle Construction Theorem, and reason ⑤ is the statement just established. Through a point not on a line, there is exactly one line parallel to that line.

Theorem 13-7: If two parallel lines are cut by a transversal, then corresponding angles are congruent.

Given: $l \parallel m$

Prove: $\angle 1 \cong \angle 2$



Argument

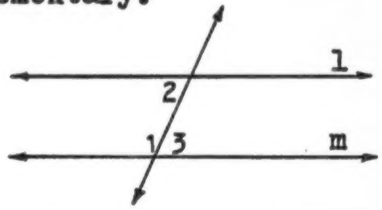
$$\text{①}\ l \parallel m \xrightarrow{\text{②}} \left.\begin{array}{l} \angle 1 \cong \angle 3 \\ \text{③}\ \angle 3 \cong \angle 2 \end{array}\right\} \xrightarrow{\text{④}} \angle 1 \cong \angle 2$$

$\therefore\ l \parallel m \longrightarrow \angle 1 \cong \angle 2$

Theorem 13-8: If two parallel lines are cut by a transversal, then consecutive interior angles are supplementary.

Given: $l \parallel m$

Prove: $\angle 1$ suppl. $\angle 2$



Argument

$$\text{①}\ l \parallel m \xrightarrow{\text{②}} \left.\begin{array}{l} m\angle 2 = m\angle 3 \\ \text{③}\ m\angle 1 + m\angle 3 = 180 \end{array}\right\} \xrightarrow{\text{④}} m\angle 1 + m\angle 2 = 180 \xrightarrow{\text{⑤}} \angle 1 \text{ suppl. } \angle 2$$

$\therefore\ l \parallel m \longrightarrow \angle 1$ suppl. $\angle 2$

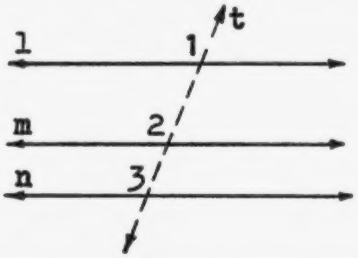
If you made the generalization suggested following theorem 13-4, you may now speculate on this one.

If two parallel lines are cut by a transversal, then any transversal pair of angles are either congruent or supplementary.

Theorem 13-9: In a plane, if two lines are parallel to a third line, then they are parallel to each other.

Given: l, m, n coplanar
$m \parallel l$, $n \parallel l$

Prove: $m \parallel n$



Argument

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① Let t be a transversal of l, m, and n.

$$\left.\begin{array}{l} \text{②}\ m \parallel l \xrightarrow{\text{③}} \angle 1 \cong \angle 2 \\ \text{④}\ n \parallel l \xrightarrow{\text{⑤}} \angle 1 \cong \angle 3 \end{array}\right\} \xrightarrow{\text{⑥}} \angle 2 \cong \angle 3 \xrightarrow{\text{⑦}} m \parallel n$$

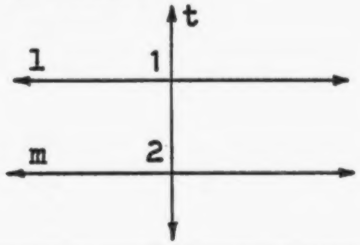
$$\therefore \left.\begin{array}{l} m \parallel l \\ n \parallel l \end{array}\right\} \longrightarrow m \parallel n$$

The theorem is also true if l, m, and n are not coplanar, but we cannot prove it with the reasons available at this time.

Theorem 13-10: In a plane, if a line is perpendicular to one of two parallel lines, it is perpendicular to the other.

Given: l, m, t coplanar
$t \perp l$, $l \parallel m$

Prove: $t \perp m$



Argument

$$\left.\begin{array}{l} \text{①}\ t \perp l \xrightarrow{\text{②}} \angle 1 \text{ is a rt. } \angle \\ \qquad \text{③}\ l \parallel m \xrightarrow{\text{④}} \angle 1 \cong \angle 2 \end{array}\right\} \longrightarrow \ldots \xrightarrow{\text{⑤}} \angle 2 \text{ is a rt. } \angle \xrightarrow{\text{⑥}} t \perp m$$

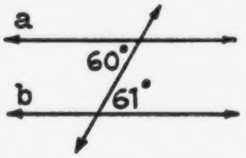
$$\therefore \left.\begin{array}{l} t \perp l \\ l \parallel m \end{array}\right\} \longrightarrow t \perp m$$

EXERCISE SET 13.3

1. Prove: If two parallel lines are cut by a transversal, alternate exterior angles are congruent.

2. Prove: If two parallel lines are cut by a transversal, consecutive exterior angles are supplementary.

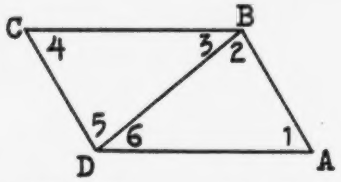
3. Write the contrapositive of theorem 13-6. What does it allow you to conclude about lines a and b in the figure?



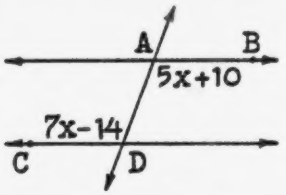
Complete each implication by stating the angles that are congruent.

4. $\overline{AB} \parallel \overline{DC} \longrightarrow$

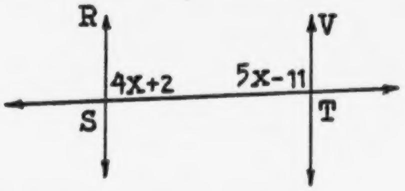
5. $\overline{AD} \parallel \overline{BC} \longrightarrow$



6. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, find x, m∠BAD, and m∠ADC.



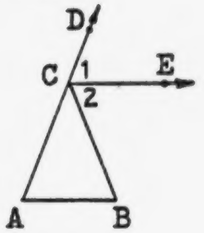
7. If $\overleftrightarrow{RS} \parallel \overleftrightarrow{VT}$, find x, m∠RST, and m∠VTS.



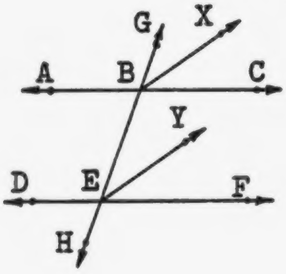
8. Write the statements below in flow proof form.

Given: $\angle 1 \cong \angle 2$, $\overrightarrow{CE} \parallel \overline{AB}$
Prove: $\overline{AC} \cong \overline{BC}$

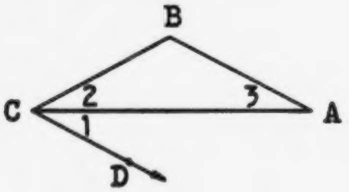
a) $\angle 1 \cong \angle 2$
b) $\angle A \cong \angle 2$
c) $\overline{AC} \cong \overline{BC}$
d) $\angle 1 \cong \angle A$
e) $\overrightarrow{CE} \parallel \overline{AB}$
f) $\angle 2 \cong \angle B$
g) $\angle A \cong \angle B$



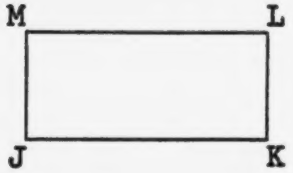
9. Given: $\overleftrightarrow{AC} \parallel \overleftrightarrow{DF}$
$\overrightarrow{BX}$ bisects ∠GBC
$\overrightarrow{EY}$ bisects ∠BEF
Prove: $\overrightarrow{BX} \parallel \overrightarrow{EY}$



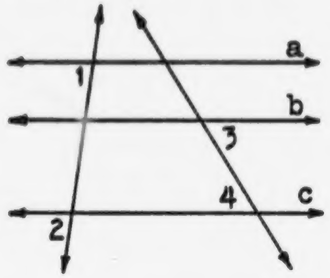
10. Given: $\overline{CA}$ bisects $\angle BCD$
$\overline{BA} \parallel \overline{CD}$
Prove: $\angle 2 \cong \angle 3$



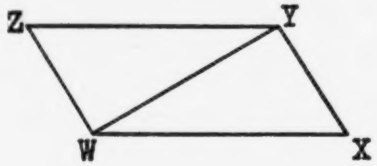
11. Given: $\overline{JK} \parallel \overline{ML}$
$\overline{JK} \perp \overline{KL}$, $\overline{JM} \perp \overline{ML}$
Prove: $\overline{JM} \parallel \overline{KL}$



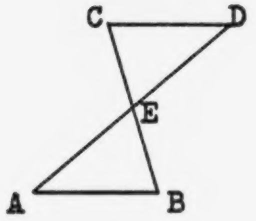
12. Given: $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$
Prove: $a \parallel b$



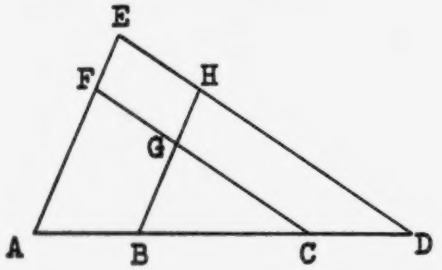
13. Given: $\overline{WX} \cong \overline{YZ}$, $\overline{WX} \parallel \overline{ZY}$
Prove: $\overline{WZ} \parallel \overline{XY}$



14. Given: $\overline{AB} \parallel \overline{CD}$, $\overline{AB} \cong \overline{DC}$
Prove: $\overline{AD}$ and $\overline{BC}$ bisect each other



15. Given: $AB = CD$, $\angle E \cong \angle FGH$
$\angle ACF \cong \angle D$
Prove: $\triangle ACF \cong \triangle BDH$

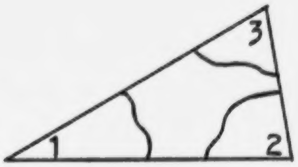


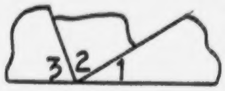
16. Draw a large scalene triangle. Label it $\triangle ABC$. Construct the angle bisector of $\angle A$ to intersect $\overline{BC}$ at D. Construct the perpendicular bisector of $\overline{AD}$ to intersect $\overline{AC}$ at E. Draw $\overline{ED}$. Using what you have constructed as the "given", prove $\overline{ED} \parallel \overline{AB}$.

Parallels Applied to Triangles

Somewhere in your earlier grades you may have been asked to perform the following experiment. If not, you may want to try it now.

Draw any triangle and tear off the three vertices. Place those three vertices together so that the sides coincide and observe the results. If you did this for ten differently shaped triangles, you should be able to arrive at the conclusion, inductively, that the sum of the measures of the three angles of any triangle is 180.



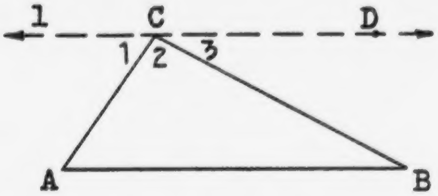


With our knowledge of parallels, we can now prove, deductively, that the statement is true. It is a very important theorem in Euclidean geometry.

Theorem 13-11: The sum of the measures of the angles of any triangle is 180.

Given: $\triangle ABC$

Prove: $m\angle A + m\angle B + m\angle C = 180$



Argument

① Draw l through C and parallel to $\overline{AB}$.
Let D be a second point on l.

$$\left.\begin{array}{l} \textcircled{2}\ m\angle 1 + m\angle ACD = 180 \\ \textcircled{3}\ m\angle 2 + m\angle 3 = m\angle ACD \end{array}\right\} \xrightarrow{\textcircled{4}} m\angle 1 + m\angle 3 + m\angle 2 = 180$$

$$\textcircled{1}\ l \parallel \overline{AB} \begin{array}{l} \xrightarrow{\textcircled{5}} m\angle 1 = m\angle A \\ \xrightarrow{\textcircled{6}} m\angle 3 = m\angle B \end{array}$$

$$\left.\begin{array}{l} m\angle 1 + m\angle 3 + m\angle 2 = 180 \\ m\angle 1 = m\angle A \\ m\angle 3 = m\angle B \end{array}\right\} \xrightarrow{\textcircled{7}}$$

$$\xrightarrow{\textcircled{7}} m\angle A + m\angle B + m\angle C = 180$$

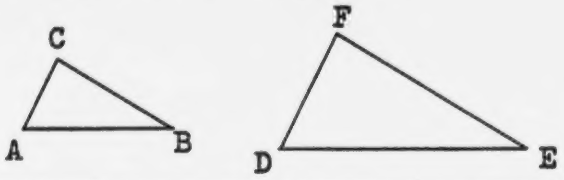
$\therefore$ For any $\triangle ABC$, $m\angle A + m\angle B + m\angle C = 180$

① Through a point not on a line, there is exactly one line parallel to that line.

As an important theorem, it has several offspring, or corollaries. We will state them and leave their proofs as exercises.

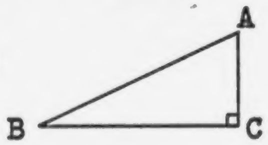
Corollary 13-11a: If two angles of one triangle are congruent to two angles of another triangle, then the third angles are congruent.

$$\left.\begin{array}{l}\angle A \cong \angle D\\ \angle B \cong \angle E\end{array}\right\} \longrightarrow \angle C \cong \angle F$$



Corollary 13-11b: The acute angles of a right triangle are complementary.

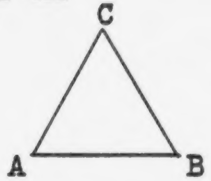
$\angle C$ is a rt. $\angle$ in $\triangle ABC \longrightarrow \angle A$ compl. $\angle B$



(note) We proved earlier that if a triangle has one right angle, the other two angles must be acute.

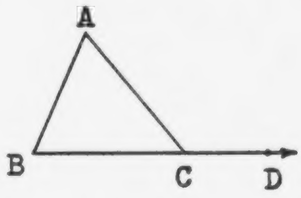
Corollary 13-11c: The measure of each angle of an equiangular triangle is 60.

$$\triangle ABC \text{ is equiangular} \begin{array}{l} \nearrow m\angle A = 60 \\ \rightarrow m\angle B = 60 \\ \searrow m\angle C = 60 \end{array}$$



Corollary 13-11d: The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

$\angle ACD$ is an ext. $\angle$ of $\triangle ABC \longrightarrow m\angle ACD = m\angle A + m\angle B$

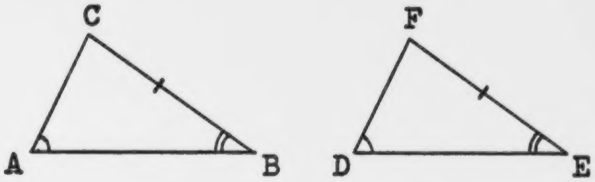


If you recall from the chapter on congruence, Postulate 12a stated that the AAS correspondence is a congruence. It was also stated that it could be proved as a theorem, but not at that time. We can now keep our word and show that it can be easily proved with the use of corollary 13-11a.

Theorem 13-12: If two angles and an unincluded side of one triangle are congruent to two angles and the corresponding unincluded side of a second triangle, the triangles are congruent. (AAS)

Given: $\angle A \cong \angle D$
$\angle B \cong \angle E$
$\overline{BC} \cong \overline{EF}$

Prove: $\triangle ABC \cong \triangle DEF$



Argument

$$\left.\begin{array}{l}\text{①}\ \angle A \cong \angle D\\ \text{②}\ \angle B \cong \angle E\end{array}\right\}\xrightarrow{\text{③}}\left.\begin{array}{l}\angle C \cong \angle F\\ \text{④}\ \overline{BC} \cong \overline{EF}\\ \text{②}\ \angle B \cong \angle E\end{array}\right\}\xrightarrow{\text{⑤}}\triangle ABC \cong \triangle DEF$$

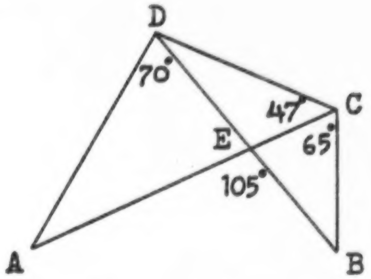
$$\therefore\ \left.\begin{array}{l}\angle A \cong \angle D\\ \angle B \cong \angle E\\ \overline{BC} \cong \overline{EF}\end{array}\right\}\longrightarrow \triangle ABC \cong \triangle DEF$$

In proving the theorem belatedly, we must be very careful that we did not use Postulate 12a in any manner whatever to eventually prove corollary 13-11a, step ③, as we would be using a statement to prove itself.

EXERCISE SET 13.4

1. Prove corollary 13-11a.
2. Prove corollary 13-11b.
3. Prove corollary 13-11c.
4. Prove corollary 13-11d.
5. Write the reasons for the statements in the proof of theorem 13-12.

6. Find the measures of all angles not given.



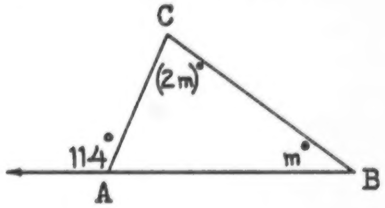
If the measures of two angles of a triangle are as follows, write the measure of the third angle.

7. 55, 108 8. 23, 35 9. x, 2x 10. 70 + a, 70 - a

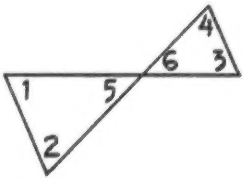
11. If the measures of the three angles of a triangle are expressed as $3x + 2$, $8x + 7$, and $10x + 3$, find x and the three measures.

12. If the measures of the three angles of a triangle are in the ratio 2:3:4, find the measure of each angle.

13. Find the measures of the three angles of ΔABC. Is it necessary to find the measure of the angle at A before finding the measures at B and C?



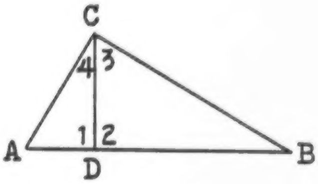
14. Given: $\angle 1 \cong \angle 3$
Prove: $\angle 2 \cong \angle 4$



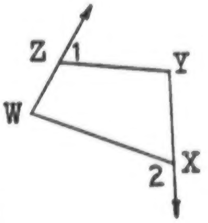
15. Write the statements below in flow proof form.

Given: $\overline{AC} \perp \overline{CB}$, $\overline{CD} \perp \overline{AB}$
Prove: $\angle A \cong \angle 3$

a) $\overline{AC} \perp \overline{CB}$
b) $\overline{CD} \perp \overline{AB}$
c) $\angle A \cong \angle 3$
d) $\angle A$ compl. $\angle 4$
e) $\angle 1$ is a rt. $\angle$ of ΔACD
f) $\angle 3$ compl. $\angle 4$

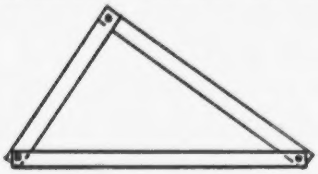


16. Given: The figure as shown
Prove: $m\angle 1 + m\angle 2 = m\angle W + m\angle Y$
(Hint: draw $\overline{WY}$)



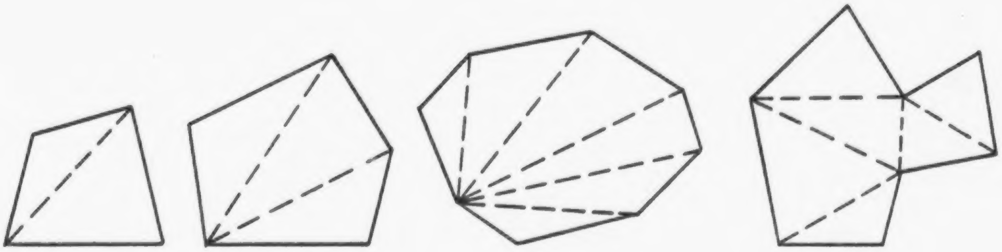
Polygons

So far, the only closed figures we have studied have been triangles. The triangle is one of a class of closed figures with segments for sides called polygons, and of course, it is the one with the fewest number of sides. We spend considerable time studying the triangle because it is a very special figure. In a physical sense it is called a rigid figure. That is, if three boards were joined with pins as shown, so that each joint were free to rotate if it could, the figure would not change its shape as a result of a small pressure at any joint.



This is actually a restatement of the SSS congruence postulate. But the same thing would not be true if there were more than three boards. A small pressure at any joint would begin to collapse the figure. This is why the triangle occurs so frequently in construction where rigidity is required, such as bridges and trusses.

Another important reason for studying the triangle so thoroughly is that figures with more sides can be divided into triangles by drawing as many additional segments as necessary. The figures below show how this can be done in various ways. This allows us to study these other figures by using what we already know about triangles.

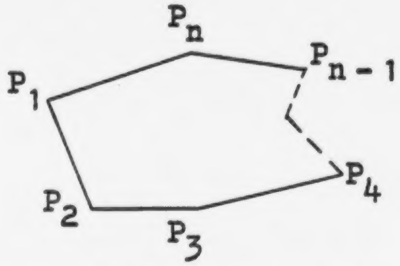


You already have a feeling for what a polygon is, but we will now give a formal definition.

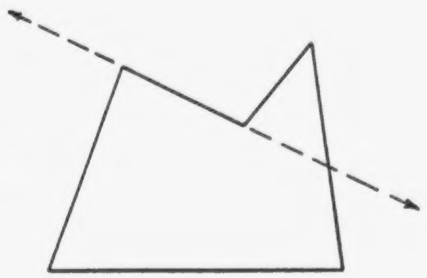
If P_1, P_2, ..., P_n are n distinct points in a plane with $n \geq 3$, then the union of the segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, ..., $\overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ is a polygon under the following conditions.

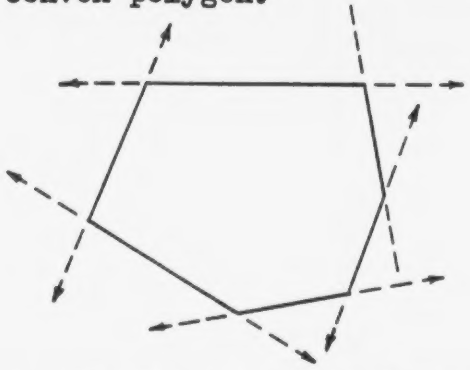
1. No two of the segments intersect except at their endpoints.
2. No two segments with a common endpoint are collinear.

The points P_1, P_2, ..., P_n are the vertices of the polygon and the segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, ..., $\overline{P_{n-1}P_n}$, $\overline{P_nP_1}$ are the sides.



You will notice that the definition allows a polygon to have "dents" in it, and we want to distinguish between those that do and those that do not. The polygon on the left below shows a line containing a side of the polygon with points of the polygon in opposite half planes with respect to this line. For the polygon on the right, no such line exists. The one on the right is a convex polygon. The one on the left is not a convex polygon.

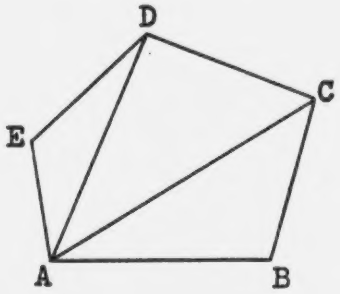




A polygon is convex if and only if no two points of it lie in opposite half planes with respect to any line that contains a side.

All of our discussion of polygons will concern only convex polygons, so from now on we will agree that the word polygon will mean convex polygon unless stated otherwise.

The two endpoints of any one side of a polygon are called consecutive vertices. A polygon is named by writing its vertices in consecutive order such as polygon ABCDE. Any two sides which intersect are called consecutive sides. A segment which joins two nonconsecutive vertices is a diagonal. $\overline{AC}$ and $\overline{AD}$ are diagonals from vertex A. The angles of a polygon, its interior and exterior are defined in a manner similar to those for a triangle.

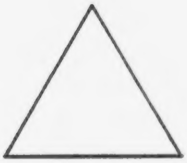


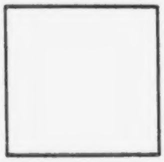
Polygons are named according to the number of sides. Some of these names are given below.

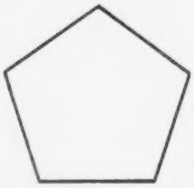
3 sides: triangle
4 sides: quadrilateral
5 sides: pentagon
6 sides: hexagon
8 sides: octagon
10 sides: decagon
12 sides: dodecagon

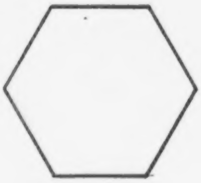
The generalized term for a polygon of n sides is n-gon. So a polygon of 17 sides is a 17-gon.

The polygons that you probably see most often are those that look the same as you rotate them to look at them from each side. These are called regular polygons. Four common examples are shown here.



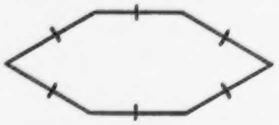




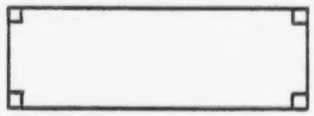


A regular polygon is a polygon that is equilateral and equiangular.

We already have the statement that an equilateral triangle is also equiangular, but this is not true for other polygons.



equilateral but
not equiangular



equiangular but
not equilateral

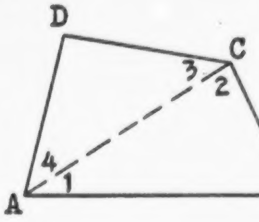
Neither of these is a regular polygon because they do not satisfy both conditions in the definition.

As mentioned earlier in this section, we can now consider some theorems about polygons that make use of our knowledge of triangles.

Theorem 13-13: The sum of the measures of the angles of any quadrilateral is 360.

Given: Quad. ABCD

Prove: $m\angle A + m\angle B + m\angle C + m\angle D = 360$



Argument

① Draw $\overline{AC}$

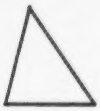
$$\left.\begin{array}{l} \text{②}\ m\angle 1 + m\angle 2 + m\angle B = 180 \\ \text{③}\ m\angle 4 + m\angle 3 + m\angle D = 180 \end{array}\right\} \xrightarrow{\text{④}} \left.\begin{array}{r} m\angle 1 + m\angle 4 + m\angle 2 + m\angle 3 + m\angle B + m\angle D = 360 \\ \text{⑤}\ m\angle 1 + m\angle 4 = m\angle BAD \\ \text{⑥}\ m\angle 2 + m\angle 3 = m\angle BCD \end{array}\right\} \xrightarrow{\text{⑦}}$$

$$\xrightarrow{\text{⑦}} m\angle BAD + m\angle BCD + m\angle B + m\angle D = 360$$

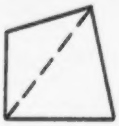
$\therefore$ For any quad. ABCD, $m\angle A + m\angle B + m\angle C + m\angle D = 360$

The fact that any polygon can be divided into triangles by drawing all diagonals from one vertex would allow us to use a similar approach for determining the sum of the measures of the angles of a polygon with any number of sides. But let's use an inductive approach to determine a formula for the sum of the measures of the angles of any n-gon in general.

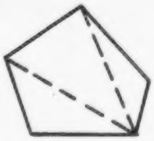
Would you agree that in each of the following figures, the sum of the measures of the angles of all the triangles could be proved equal to the sum of the measures of the angles of the polygon?



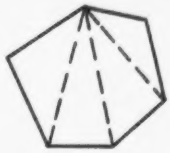
3 sides
1 Δ



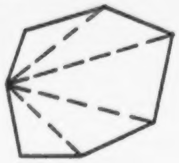
4 sides
2 Δs



5 sides
3 Δs



6 sides
4 Δs



7 sides
5 Δs

Get the pattern? As the number of sides is increased by one, the number of triangles is also increased by one, and the number of triangles is always two less than the number of sides. Since we know the sum of the measures of the angles of one triangle is 180, we can simply multiply 180 by the number of triangles.

Theorem 13-14: The sum of the measures of the angles of any convex n-gon is $180(n-2)$.

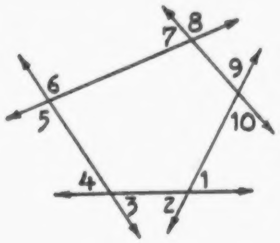
We will not consider a formal proof of the general case as it involves techniques not available to us at this time.

In a regular polygon, all the angles have equal measure, so we can use theorem 13-14 to find the measure of one of them. Doesn't the number of angles in any polygon equal the number of sides? If we know that the sum of the measures of all the angles of a regular n-gon is $180(n-2)$, we can simply divide by the number of angles to find the measure of one angle.

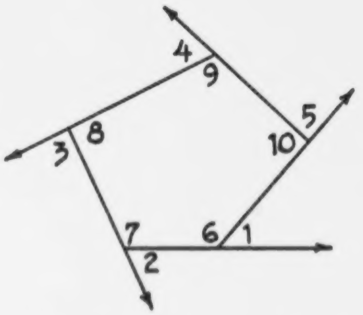
Corollary 13-14a: The measure of one angle of a regular n-gon is $\frac{180(n-2)}{n}$.

When we have referred to the angles of a polygon, we have understood this to mean the interior angles. Just as with triangles, other polygons have two exterior angles at each vertex.

In the pentagon shown, we can see 10 exterior angles in five congruent pairs.



If we consider the sum of the measures of the five exterior angles of a pentagon, one at each vertex, we will number the exterior and adjacent interior angles as shown. Since the ten angles consist of five linear pairs, we can write the equation below.



$$(m\angle 1+m\angle 6)+(m\angle 2+m\angle 7)+(m\angle 3+m\angle 8)+(m\angle 4+m\angle 9)+(m\angle 5+m\angle 10) = 180\cdot 5$$

But $m\angle 6 + m\angle 7 + m\angle 8 + m\angle 9 + m\angle 10 = 180(5-2) = 180\cdot 3$

By subtraction we have

$$m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 + m\angle 5 = 180\cdot 5 - 180\cdot 3 = 180\cdot 2 = 360$$

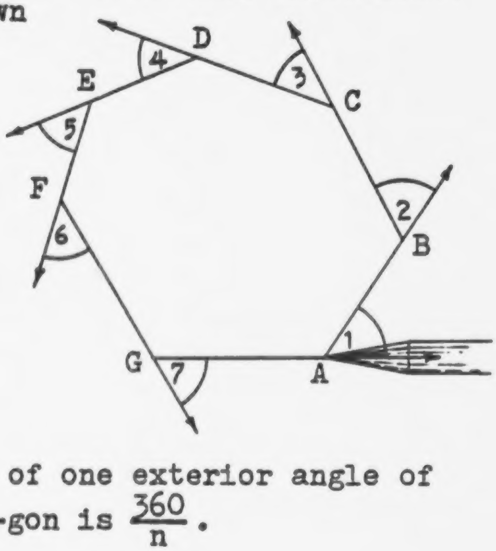
Now if we generalize this procedure for any n-gon, we would subtract the sum of the measures of the n interior angles from the sum of the measures of the n linear pairs to get the sum of the measures of the n exterior angles.

$$180\cdot n - 180(n-2) = 180n - 180n + 360 = 360$$

Theorem 13-15: The sum of the measures of the exterior angles of any convex n-gon, one at each vertex, is 360.

The theorem says that the sum of the measures of these exterior angles is independent of the number of sides. If this seems strange to you, try the following experiment.

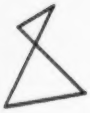
Locate your pencil as shown with the point at vertex A. Rotate the pencil through $\angle 1$ so it now lies on $\overline{AB}$. Slide the pencil point to vertex B. Rotate through $\angle 2$. Continue this procedure until your pencil is where it started. Haven't you rotated your pencil one full turn, or 360°?

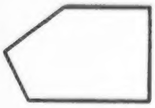


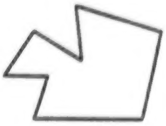
Corollary 13-15a: The measure of one exterior angle of a regular n-gon is $\frac{360}{n}$.

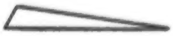
EXERCISE SET 13.5

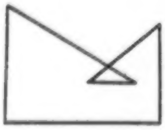
1. Identify each figure below in which the union of the segments is a polygon.
2. Identify those which are convex polygons.

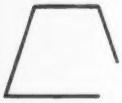

a


b


c


d


e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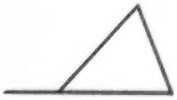
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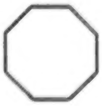
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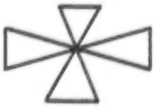
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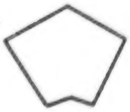

k

3. Name the polygon with the number of sides as follows.

   a) 8 c) 7 d) 22 g) 6
   b) 5 d) 3 f) 4 h) 10

4. Is there any polygon that can be classified as a regular polygon just because it is equilateral?

If the measures of three angles of a quadrilateral are as follows, write the measure of the fourth angle.

5. 75, 40, 70
6. 85, 85, 85
7. 179, 91, 89
8. x, 2x, 3x

9. Can a quadrilateral have four acute angles?
10. What is the largest number of obtuse angles a quadrilateral may have?
11. If the measures of the four angles of a quadrilateral are in the ratio 2:3:5:8, find the measure of each.
12. If three angles of a quadrilateral have equal measures, then the measure of each of the three must be less than ? and more than ?.
13. If all diagonals are drawn from one vertex of a convex 15-gon, how many triangles are formed? What is the sum of the measures of the angles of a 15-gon?

14. Find the sum of the measures of the angles of each of the following.

    a) pentagon b) decagon c) hexagon d) octagon

15. How many sides does a convex polygon have if the sum of the measures of its angles is 1980?

16. Find the sum of the measures of the exterior angles of each of the following.

    a) octagon b) triangle c) decagon d) pentagon

17. What is another name for a regular quadrilateral.

18. Find the measure of one interior angle of each of the following.

    a) regular hexagon
    b) regular decagon
    c) regular octagon
    d) regular pentagon
    e) regular 7-gon
    f) regular 20-gon

19. If the measure of one interior angle of a regular n-gon is 150, find the number of sides.

20. Find the measure of one exterior angle of each of the following.

    a) regular hexagon
    b) regular pentagon
    c) regular triangle
    d) regular 15-gon

21. Prove: The sum of the measures of the angles of a pentagon is 540.

22. A formula for the number of diagonals of any n-gon can be discovered by induction. Examine the figures on page 166 to determine the relationship between the number of diagonals from one vertex and the number of sides.

    How many diagonals could be drawn from one vertex of an octagon? of a 14-gon? of an n-gon?

    If you count all the diagonals from every vertex, you would be counting each diagonal twice. Find the number of diagonals of an octagon without counting. Find the number of diagonals of a 14-gon.

    Write a formula for the number of diagonals of any n-gon.

CHAPTER 14 - QUADRILATERALS
or FOUR ON THE FLOOR

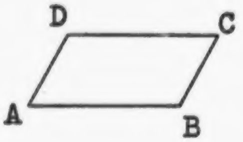
Definitions

Probably the most familiar geometric figure to you is the rectangle. In fact, you're looking at one right now. The rectangle is just one of several special quadrilaterals with many special and interesting properties. The following definitions show their relationships to one another.

A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

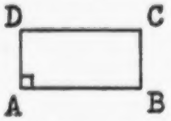
The symbol for parallelogram is ▱

$$\left.\begin{array}{r}\overline{AB}\parallel\overline{DC}\\ \overline{AD}\parallel\overline{BC}\end{array}\right\}\longleftrightarrow \text{ABCD is a ▱}$$



A rectangle is a parallelogram with a right angle.

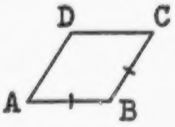
$$\left.\begin{array}{r}\text{ABCD is a ▱}\\ \angle A \text{ is a rt.} \angle\end{array}\right\}\longleftrightarrow \text{ABCD is a rectangle}$$



It can be shown from this definition that all four angles must be right angles as you know.

A rhombus is a parallelogram with two consecutive sides congruent.

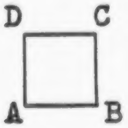
$$\left.\begin{array}{r}\text{ABCD is a ▱}\\ \overline{AB}\cong\overline{BC}\end{array}\right\}\longleftrightarrow \text{ABCD is a rhombus}$$



It can be shown that all four sides of a rhombus are congruent.

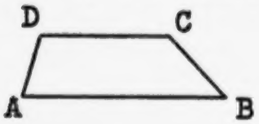
A square is both a rectangle and a rhombus.

$$\left.\begin{array}{r}\text{ABCD is a rectangle}\\ \text{ABCD is a rhombus}\end{array}\right\}\longleftrightarrow \text{ABCD is a square}$$



Other definitions could be accepted for the rectangle, rhombus, and square. By giving the definitions in this way we are emphasizing the fact that when a theorem is proved for a parallelogram it is also proved for the rectangle, rhombus, and square. When a theorem is proved about a rectangle or rhombus, it is also proved for a square.

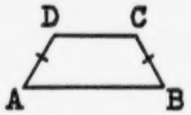
A trapezoid is a quadrilateral with exactly one pair of parallel sides.



$\left.\begin{array}{l}\overline{AB}\parallel\overline{DC}\\ \overline{AD}\nparallel\overline{BC}\end{array}\right\}\longleftrightarrow$ ABCD is a trapezoid

The parallel sides, $\overline{AB}$ and $\overline{DC}$, are the bases.
The nonparallel sides, $\overline{AD}$ and $\overline{BC}$, are the legs.
Two angles that share a base as a side are base angles.
∠A and ∠B are base ∠s, ∠C and ∠D are base ∠s.

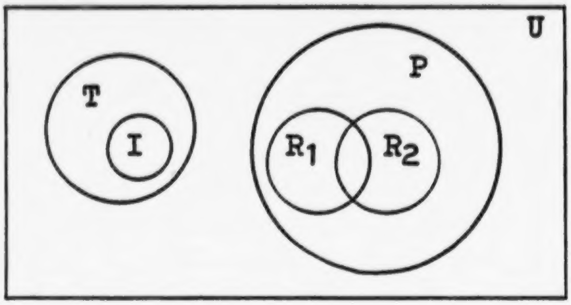
An isosceles trapezoid is a trapezoid with congruent legs.



$\left.\begin{array}{l}\overline{AB}\parallel\overline{DC}\\ \overline{AD}\cong\overline{BC}\end{array}\right\}\longleftrightarrow$ ABCD is an isosceles trapezoid

The relationships of quadrilaterals can be shown very nicely with a Venn diagram as illustrated below.

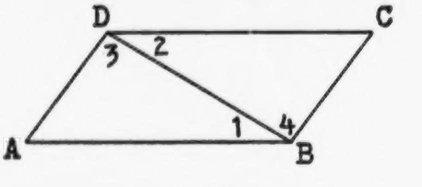
U = {quadrilaterals}
P = {parallelograms}
R_1 = {rectangles}
R_2 = {rhombuses}
T = {trapezoids}
I = {isos. trapezoids}
$R_1 \cap R_2$ = {squares}



Theorems About Parallelograms

Theorem 14-1: A diagonal of a parallelogram separates the parallelogram into two congruent triangles.

Given: ABCD is a ▱
Prove: △ABD ≅ △CDB



Argument

$$\begin{tikzcd}[row sep=small]
 & \overline{AB}\parallel\overline{DC} \arrow[r, "④"] & \angle 1\cong\angle 2 & \\
\text{① ABCD is a ▱} \arrow[ur, "②"] \arrow[r, "③"] & \overline{AD}\parallel\overline{BC} \arrow[r, "⑤"] & \angle 3\cong\angle 4 \arrow[r, "⑦"] & \triangle ABD\cong\triangle CDB \\
 & & \text{⑥ } \overline{BD}\cong\overline{DB} &
\end{tikzcd}$$

∴ ABCD is a ▱ ⟶ △ABD ≅ △CDB

It is actually the corollaries of this theorem that provide us with the properties we are interested in.

Corollary 14-1a: Opposite sides of a parallelogram are congruent.

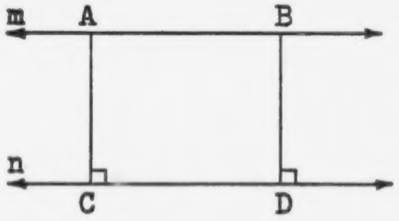
$$\text{ABCD is a } ▱ \begin{array}{l} \nearrow \overline{AB} \cong \overline{DC} \\ \searrow \overline{AD} \cong \overline{BC} \end{array}$$

Corollary 14-1b: Opposite angles of a parallelogram are congruent.

$$\text{ABCD is a } ▱ \begin{array}{l} \nearrow \angle A \cong \angle C \\ \searrow \angle B \cong \angle D \end{array}$$

The next corollary involves the idea of distance from a point to a line. This has already been defined as the length of the perpendicular segment from the point to the line.

Corollary 14-1c: If two lines are parallel, all points of each line are equidistant from the other line.



Given: $m \parallel n$
A and B are any two points on m
$\overline{AC} \perp n$, $\overline{BD} \perp n$

Prove: $AC = BD$

Argument

$$\left.\begin{array}{l} \text{①}\ \overline{AC} \perp n \\ \text{②}\ \overline{BD} \perp n \end{array}\right\} \xrightarrow[\text{④}]{\text{③}} \left.\begin{array}{l} \overline{AC} \parallel \overline{BD} \\ \overline{AB} \parallel \overline{CD} \end{array}\right\} \xrightarrow{\text{⑤}} \text{ACDB is a } ▱ \xrightarrow{\text{⑥}} AC = BD$$

$\therefore m \parallel n \longrightarrow$ A and B are equidistant from n

---

This corollary is frequently stated as follows.

Parallel lines are everywhere equidistant.

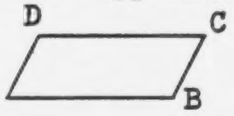
This allows us to make the following definition.

The distance between two parallel lines is the length of a perpendicular segment from any point on one line to the other line.

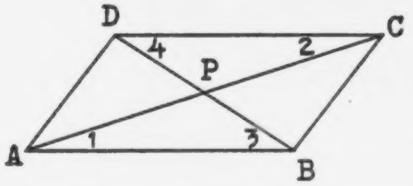
Theorem 13-8 about parallel lines can be stated as a theorem about parallelograms.

Consecutive angles of a parallelogram are supplementary.

ABCD is a ▱ ⟶ ∠A suppl. ∠B



Theorem 14-2: The diagonals of a parallelogram bisect each other.



Given: ABCD is a ▱

Prove: $\overline{AP} \cong \overline{CP}$, $\overline{BP} \cong \overline{DP}$

Argument

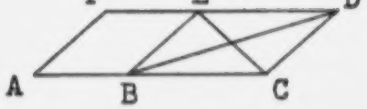
$$\begin{tikzcd}
\text{① ABCD is a ▱} \arrow[rr, "\text{②}"] \arrow[dr, "\text{③}"'] & & \overline{AB} \cong \overline{CD} \arrow[dr] & & \\
& \overline{AB} \parallel \overline{DC} \arrow[r, "\text{④}"] \arrow[dr, "\text{⑤}"'] & \angle 1 \cong \angle 2 \arrow[r, "\text{⑥}"] & \triangle ABP \cong \triangle CDP \arrow[r, "\text{⑦}"] \arrow[r, shift right=2, "\text{⑧}"'] & {} \\
& & \angle 3 \cong \angle 4 \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\text{⑦}"] & \overline{AP} \cong \overline{CP} \\
{} \arrow[r, "\text{⑧}"] & \overline{BP} \cong \overline{DP}
\end{tikzcd}$$

$$\begin{tikzcd}
\therefore \text{ ABCD is a ▱} \arrow[r] \arrow[dr] & \overline{AP} \cong \overline{CP} \\
& \overline{BP} \cong \overline{DP}
\end{tikzcd}$$

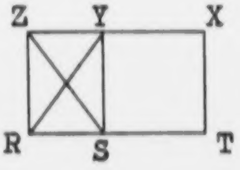
EXERCISE SET 14.1

$\overline{AC} \parallel \overline{FD}$, $\overline{AF} \parallel \overline{BE} \parallel \overline{CD}$, $\overline{AF} \cong \overline{FE} \cong \overline{EC}$



1. Name a rhombus.
2. Name a parallelogram that is not a rhombus.
3. Name an isosceles trapezoid.
4. Name a trapezoid that is not isosceles.

$\overline{RT} \parallel \overline{ZX}$, $\overline{RZ} \parallel \overline{SY} \parallel \overline{TX}$, $\overline{TX} \cong \overline{XY}$, $\overline{XT} \perp \overline{RT}$



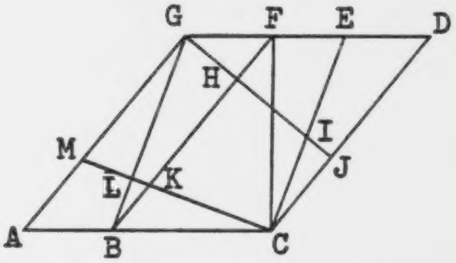
5. Name a square.
6. Name a rectangle that is not a square.
7. Name a parallelogram.
8. Name a rhombus.
9. Name two trapezoids.

10. Prove corollary 14-1a.

11. Prove corollary 14-1b.

$\overline{AC} \parallel \overline{GD}$, $\overline{AG} \parallel \overline{BF} \parallel \overline{CD}$, $\overline{BG} \parallel \overline{CE}$, $\overline{GJ} \perp \overline{CD}$, $\overline{CM} \perp \overline{BG}$, $\overline{FC} \perp \overline{GD}$
Write the distance between each pair of parallel segments.

12. $\overline{AC}$ and $\overline{GD}$
13. $\overline{BF}$ and $\overline{CD}$
14. $\overline{BG}$ and $\overline{CE}$
15. $\overline{AG}$ and $\overline{BF}$

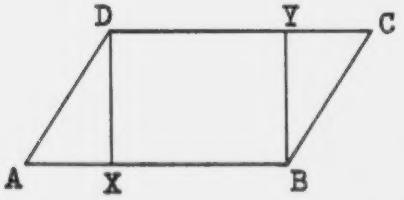


16. Write the reasons for the statements in the proof of theorem 14-2.

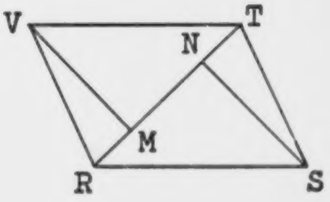
17. Prove: All four angles of a rectangle are right angles.

18. Prove: All four sides of a rhombus are congruent.

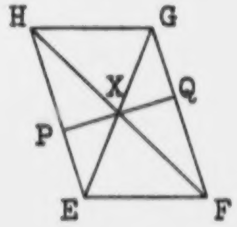
19. Given: ABCD is a ▱
$\overline{DX} \perp \overline{AB}$, $\overline{BY} \perp \overline{DC}$
Prove: $\overline{AX} \cong \overline{CY}$



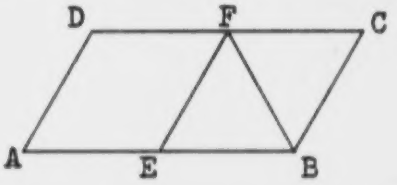
20. Given: RSTV is a ▱
$\overline{VM} \perp \overline{RT}$, $\overline{SN} \perp \overline{RT}$
Prove: $\overline{VM} \cong \overline{SN}$



21. Given: EFGH is a ▱
Prove: $\overline{PX} \cong \overline{QX}$



22. Given: ABCD is a ▱, m∠A = 60
AEFD is a rhombus
ABFD is an isos. trap.
Prove: EBCF is a rhombus



Some earlier induction exercises relate closely to the next section.

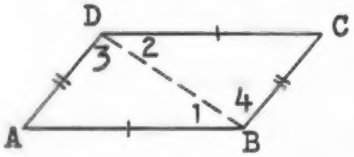
23. Rework exercise 5 from exercise set 3.1.

24. Rework exercise 7 from exercise set 3.1.

Proving That Quadrilaterals are Parallelograms

The only way we have at present to justify that a quadrilateral is a parallelogram is by the definition, that both pairs of opposite sides sre parallel. This will allow us to prove the converses of the earlier theorems and corollaries of this chapter. We will show the proof of the first one.

Theorem 14-3: If both pairs of opposite sides of a quadrilateral are congruent, then it is a parallelogram.



Given: $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$

Prove: ABCD is a ▱

Argument

① Draw $\overline{BD}$

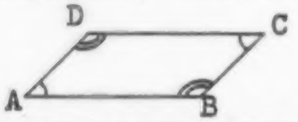
$$\left.\begin{array}{l} \text{②}\ \overline{AB} \cong \overline{CD} \\ \text{③}\ \overline{AD} \cong \overline{CB} \\ \text{④}\ \overline{BD} \cong \overline{DB} \end{array}\right\} \xrightarrow{\text{⑤}} \triangle ABD \cong \triangle CDB \begin{array}{l} \xrightarrow{\text{⑥}} \angle 1 \cong \angle 2 \xrightarrow{\text{⑧}} \overline{AB} \parallel \overline{DC} \\ \xrightarrow{\text{⑦}} \angle 3 \cong \angle 4 \xrightarrow{\text{⑨}} \overline{AD} \parallel \overline{BC} \end{array} \xrightarrow{\text{⑩}}$$

$$\xrightarrow{\text{⑩}} \text{ABCD is a ▱}$$

$$\therefore \left.\begin{array}{l} \overline{AB} \cong \overline{DC} \\ \overline{AD} \cong \overline{BC} \end{array}\right\} \longrightarrow \text{ABCD is a ▱}$$

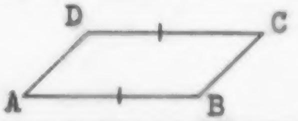
Theorem 14-4: If both pairs of opposite angles of a quadrilateral are congruent, then it is a parallelogram.

$$\left.\begin{array}{l} \angle A \cong \angle C \\ \angle B \cong \angle D \end{array}\right\} \longrightarrow \text{ABCD is a ▱}$$



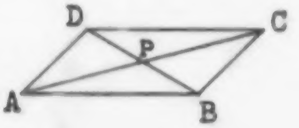
Theorem 14-5: If two sides of a quadrilateral are both parallel and congruent, then it is a parallelogram.

$$\left.\begin{array}{l} \overline{AB} \parallel \overline{DC} \\ \overline{AB} \cong \overline{DC} \end{array}\right\} \longrightarrow \text{ABCD is a ▱}$$



Theorem 14-6: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

$$\left.\begin{array}{l} \overline{AP} \cong \overline{CP} \\ \overline{BP} \cong \overline{DP} \end{array}\right\} \longrightarrow \text{ABCD is a ▱}$$

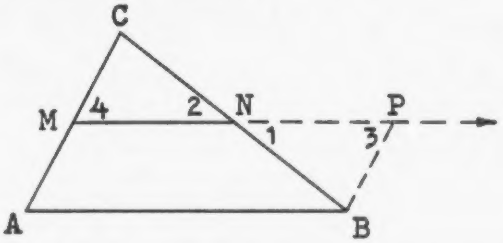


All the theorems in this chapter so far should be readily acceptable to you by intuition and without the need for proof. The next theorem is not quite so obvious. It has a compound conclusion and its proof is the longest we have presented this far. You would not be expected to initiate such a proof yet by yourself, but a careful study of it should help you in several ways. The organization of statements in a flow proof requires careful planning and perhaps a lot of scratch paper to make sure that all the implications are clearly shown. The introduction and use of auxiliary lines is frequently a mystery to students who wonder how they are supposed to know when and where to use them. The key words in initiating a difficult proof are experience, insight, and ingenuity. The study of this theorem becomes part of your experience and hopefully your insight and ingenuity will eventually allow you to prove theorems as difficult.

Theorem 14-7: The segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.

Given: M is midpoint of $\overline{AC}$
N is midpoint of $\overline{BC}$

Prove: $\overline{MN} \parallel \overline{AB}$
$MN = \frac{1}{2} AB$



Argument

① Extend $\overline{MN}$ beyond N.
② Let P be a point on $\overrightarrow{MN}$ such that NP = NM.
③ Draw $\overline{PB}$

$$\begin{tikzcd}
 & & \angle 3 \cong \angle 4 \arrow[r, "\textcircled{10}"] & \overline{MA} \parallel \overline{PB} \arrow[dr] & \\
\left.\begin{array}{l} \textcircled{2}\ \overline{NP} \cong \overline{NM} \\ \textcircled{4}\ \angle 1 \cong \angle 2 \\ \textcircled{5}\ \overline{NB} \cong \overline{NC} \end{array}\right\} \arrow[r, "\textcircled{6}"] & \triangle NBP \cong \triangle NCM \arrow[ur, "\textcircled{7}"] \arrow[dr, "\textcircled{8}"'] & & & \arrow[r, "\textcircled{12}"] & {} \\
 & & \left.\begin{array}{l} \overline{PB} \cong \overline{MC} \\ \textcircled{9}\ \overline{MA} \cong \overline{MC} \end{array}\right\} \arrow[r, "\textcircled{11}"] & \overline{MA} \cong \overline{PB} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\arrow[r, "\textcircled{12}"] & \text{MABP is a } \square \arrow[r, "\textcircled{13}"] \arrow[dr, "\textcircled{14}"] & \overline{MP} \parallel \overline{AB}\ \ (\overline{MN} \parallel \overline{AB}) & & \\
 & & \left.\begin{array}{r} MP = AB \\ \textcircled{15}\ MP = MN + NP \end{array}\right\} \arrow[r, "\textcircled{16}"] & \left.\begin{array}{r} MN + NP = AB \\ \textcircled{2}\ MN = NP \end{array}\right\} \arrow[r, "\textcircled{17}"] & {}
\end{tikzcd}$$

$$\xrightarrow{\textcircled{17}} MN + MN = AB \xrightarrow{\textcircled{18}} 2\,MN = AB \xrightarrow{\textcircled{19}} MN = \tfrac{1}{2} AB$$

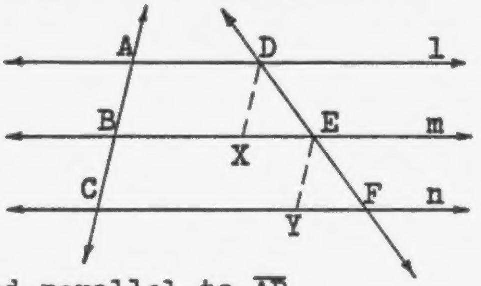
$$\therefore \left.\begin{array}{l} \text{M is the midpoint of } \overline{AC} \\ \text{N is the midpoint of } \overline{BC} \end{array}\right\} \begin{array}{l} \nearrow \overline{MN} \parallel \overline{AB} \\ \searrow MN = \tfrac{1}{2} AB \end{array}$$

The next theorem is more obviously true, but its proof is still challenging. We'll give you a start and a hint and let you take it from there.

Theorem 14-8: If three parallel lines cut off congruent segments on one transversal, they cut off congruent segments on every transversal.

Given: $l \parallel m \parallel n$
$\overline{AB} \cong \overline{BC}$

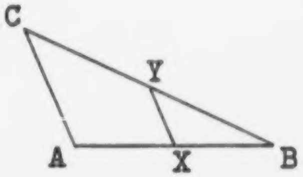
Prove: $\overline{DE} \cong \overline{EF}$



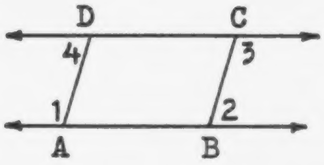
Draw a line through D and parallel to $\overline{AB}$.
Draw a line through E and parallel to $\overline{BC}$.
Call the points of intersection X and Y as shown.
Try to prove parallelograms first to help you prove $\cong \triangle$s.

EXERCISE SET 14.2

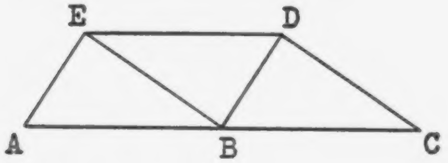
1. Write the reasons for the statements in the proof of theorem 14-3.
2. Prove theorem 14-4.
3. Prove theorem 14-5.
4. Prove theorem 14-6.
5. Restate corollary 14-1a and theorem 14-3 as a single statement in the form "p if and only if q".
6. Restate theorem 14-2 and theorem 14-6 as a single statement in the form "p if and only if q".
7. Write the reason for statement ⑫ in the proof of theorem 14-7.
8. X and Y are midpoints of $\overline{AB}$ and $\overline{BC}$.
   $XY = 7$, $m\angle B = 25$, $m\angle C = 40$
   Find AC and $m\angle AXY$.



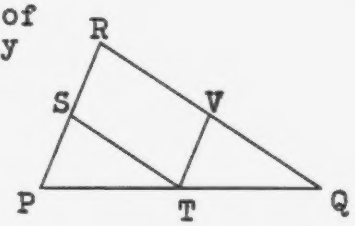
9. Given: $\angle 1 \cong \angle 3$, $\angle 2 \cong \angle 4$
Prove: ABCD is a ▱



10. Given: ABDE is a ▱, $\overline{AB} \cong \overline{BC}$
Prove: BCDE is a ▱

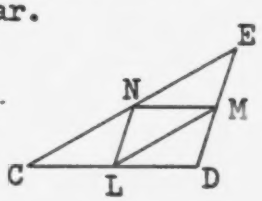


11. Given: S, T, and V are midpoints of $\overline{RP}$, $\overline{PQ}$, and $\overline{QR}$ respectively
Prove: RSTV is a ▱

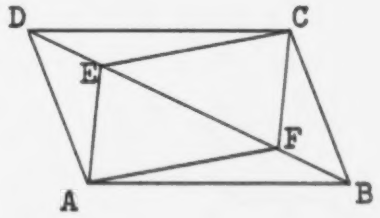


12. Prove: Bisectors of two consecutive angles of a parallelogram are perpendicular.

13. Given: L, M, and N are midpoints of $\overline{CD}$, $\overline{DE}$, and $\overline{EC}$ respectively
Prove: perimeter of ΔLMN equals half the perimeter of ΔCDE



14. Given: ABCD is a ▱, $\overline{DE} \cong \overline{BF}$
Prove: AFCE is a ▱



15. Prove: The quadrilateral formed by joining the midpoints of consecutive sides of a given quadrilateral is a parallelogram. (Recall ex. 24 from EX. SET 14.1)

16. Prove: The quadrilateral formed by joining the midpoints of consecutive sides of a rectangle is a rhombus.

17. Prove: The quadrilateral formed by joining the midpoints of consecutive sides of a rhombus is a rectangle.

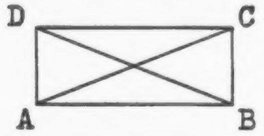
18. Prove theorem 14-8.

Theorems About the Rectangle, Rhombus, and Square

For any theorem stating a property of the parallelogram, we could substitute the word rectangle, rhombus, or square. But these quadrilaterals have additional special properties. Remember also that the square has all the properties of both the rectangle and rhombus. The theorems are generally acceptable to intuition and the proofs are straightforward.

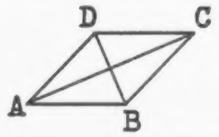
Theorem 14-9: The diagonals of a rectangle are congruent.

ABCD is a rect. $\longrightarrow \overline{AC} \cong \overline{BD}$



Theorem 14-10: The diagonals of a rhombus are perpendicular.

ABCD is a rhombus $\longrightarrow \overline{AC} \perp \overline{BD}$

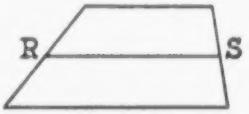


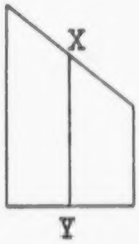
Theorems About Trapezoids

The trapezoid is not one of the more common quadrilaterals, but it does occur in very significant ways. We first give a definition in connection with a trapezoid.

The median of a trapezoid is the segment that joins the midpoints of the legs.

$\overline{RS}$ and $\overline{XY}$ are medians for the trapezoids shown





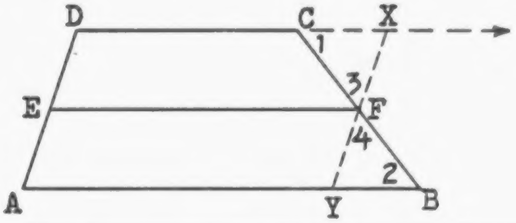
The following theorem about the median of a trapezoid has a compound conclusion and in this case we consider the proof in two parts.

Theorem 14-11: The median of a trapezoid is parallel to the bases and its length is half the sum of the lengths of the bases.

Given: $\overline{EF}$ is the median of trap. ABCD

Prove: $\overline{EF} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{DC}$

$EF = \frac{1}{2}(AB + DC)$



Argument

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① Extend $\overline{DC}$ beyond C.
② Draw a line through F and parallel to $\overline{DA}$.
③ Let X and Y be the intersections with $\overleftrightarrow{DC}$ and $\overline{AB}$.

Part I

$$\left.\begin{array}{r} \text{④}\ \overline{DC} \parallel \overline{AB} \xrightarrow{\text{⑤}} \left.\begin{array}{l} \angle 1 \cong \angle 2 \\ \text{⑥}\ \overline{CF} \cong \overline{BF} \\ \text{⑦}\ \angle 3 \cong \angle 4 \end{array}\right\} \xrightarrow{\text{⑧}} \triangle CXF \cong \triangle BYF \xrightarrow{\text{⑨}} \overline{XF} \cong \overline{YF} \rightarrow \cdots \xrightarrow{\text{⑩}} YF = \frac{1}{2} YX \\ \text{⑪}\ \overline{AE} \cong \overline{DE} \rightarrow \cdots \xrightarrow{\text{⑫}} AE = \frac{1}{2} AD \\ \left.\begin{array}{l} \text{⑬}\ \overline{DX} \parallel \overline{AY} \\ \text{②}\ \overline{AD} \parallel \overline{YX} \end{array}\right\} \xrightarrow{\text{⑭}} AYXD \text{ is a } \text{▱} \xrightarrow{\text{⑮}} YX = AD \xrightarrow{\text{⑯}} \frac{1}{2} YX = \frac{1}{2} AD \end{array}\right\} \xrightarrow{\text{⑰}}$$

$$\left.\begin{array}{r} \xrightarrow{\text{⑰}} YF = AE \\ \text{②}\ \overline{YF} \parallel \overline{AE} \end{array}\right\} \xrightarrow{\text{⑱}} YFEA \text{ is a } \text{▱} \xrightarrow{\text{⑲}} \left.\begin{array}{r} \overline{EF} \parallel \overline{AY} \\ \text{④}\ \overline{DC} \parallel \overline{AY} \end{array}\right\} \xrightarrow{\text{⑳}} \overline{EF} \parallel \overline{DC}$$

Part II

$$\left.\begin{array}{l} \text{⑱}\ YFEA \text{ is a } \text{▱} \xrightarrow{\text{㉑}} EF = AY \rightarrow \cdots \xrightarrow{\text{㉒}} EF = AB - BY \\ \text{㉓}\ EFXD \text{ is a } \text{▱} \xrightarrow{\text{㉔}} EF = DX \rightarrow \cdots \xrightarrow{\text{㉕}} EF = DC + CX \end{array}\right\} \xrightarrow{\text{㉖}}$$

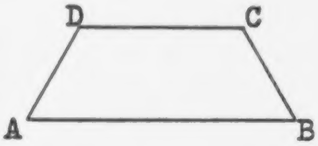
$$\left.\begin{array}{r} \xrightarrow{\text{㉖}} 2\,EF = AB - BY + DC + CX \\ \text{⑧}\ \triangle CXF \cong \triangle BYF \xrightarrow{\text{㉗}} BY = CX \end{array}\right\} \xrightarrow{\text{㉘}} 2\,EF = AB - BY + DC + BY \xrightarrow{\text{㉙}}$$

$$\xrightarrow{\text{㉙}} 2\,EF = AB + DC \xrightarrow{\text{㉚}} EF = \frac{1}{2}(AB + DC)$$

$$\therefore\ \overline{EF} \text{ is a median of trap. } ABCD \begin{array}{l} \nearrow \overline{EF} \parallel \overline{AB} \\ \rightarrow \overline{EF} \parallel \overline{DC} \\ \searrow EF = \frac{1}{2}(AB + DC) \end{array}$$

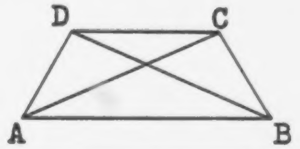
Theorem 14-12: The base angles of an isosceles trapezoid are congruent.

ABCD is an isos. trap. $\rightarrow \angle A \cong \angle B$, $\rightarrow \angle C \cong \angle D$



Theorem 14-13: The diagonals of an isosceles trapezoid are congruent.

ABCD is an isos. trap. $\longrightarrow \overline{AC} \cong \overline{BD}$

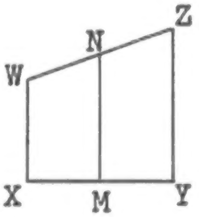


The properties of the special quadrilaterals can be easily tabulated as shown below.

	par	rect	rhom	sq	trap	isos trap
exactly one pair of opp sides ∥					x	x
exactly one pair of opp sides ≅						x
both pair of opp sides ∥	x	x	x	x		
both pair of opp sides ≅	x	x	x	x		
all four sides ≅			x	x		
both pair of opp ∠s ≅	x	x	x	x		
all ∠s are rt. ∠s		x		x		
diagonals ≅		x		x		x
diagonals ⊥			x	x		
diagonals bisect each other	x	x	x	x		
each diagonal bisects 2 ∠s			x	x		

EXERCISE SET 14.3

1. Prove theorem 14-9.
2. Prove theorem 14-10.
3. Prove: If the diagonals of a parallelogram are congruent, it is a rectangle.
4. Prove: If the diagonals of a parallelogram are perpendicular, it is a rhombus.
5. Write the reasons for statements ⑭, ⑮, ⑱, ⑲, and ⑳ in the proof of theorem 14-11.
6. $\overline{MN}$ is the median of trap. WXYZ, WX = 18, YZ = 27. Find MN.



7. Prove theorem 14-12. (Hint: Draw perpendiculars from the vertices of one base to the other base.)
8. Prove theorem 14-13.

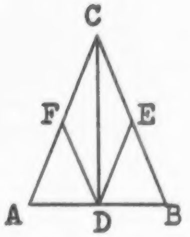
Draw a quadrilateral having each of the following conditions which is not any of the special quadrilaterals.

9. Two right angles
10. Three congruent sides
11. diagonals congruent
12. diagonals perpendicular

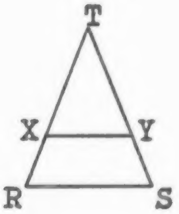
Name every special quadrilateral for which each of the following is true.

13. Both pairs of opposite sides congruent
14. Diagonals bisect each other
15. Diagonals congruent
16. All four sides congruent
17. Exactly one pair of opposite sides congruent
18. Diagonals perpendicular
19. All angles congruent
20. Each diagonal bisects two angles
21. Diagonals are perpendicular bisectors of each other

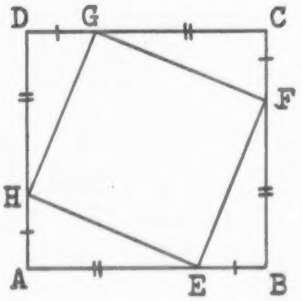
22. Given: $\overline{AC} \cong \overline{BC}$, $\overline{CD} \perp \overline{AB}$
E and F are midpoints
of $\overline{BC}$ and $\overline{AC}$
Prove: DECF is a rhombus



23. Given: RSYX is an isos. trap.
Prove: ΔRST is isosceles



24. Given: ABCD is a square and
congruent segments are
marked appropriately
Prove: EFGH is a square



CHAPTER 15 - LINES AND PLANES IN SPACE
or FAR OUT

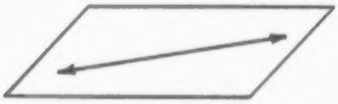
Perpendicular and Parallel Relations

Entire books have been written about the geometry of space. It is an important and interesting part of geometry, and you may enjoy making a deeper study of it than the limited presentation given here. We are necessarily restricted by the time limitations of a one year course. We will consider some basic realtionships which should appeal to your intuition and just enough proof to require you to think in terms of space.

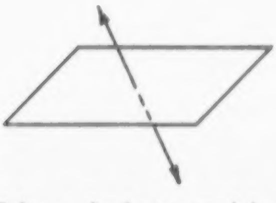
One problem that some students have with space geometry is drawing the figures. It does take practice and a few suggestions may be helpful. We cannot actually draw a plane because it has no edges, so we let a rectangle represent a plane that is infinite in extent. However, we want to draw a rectangle on this paper so that it does not look like it lies in the plane of this paper. If we draw a parallelogram as shown, it gives us the desired depth perception by thinking of the lower segment as being closer to us while we're looking at the rectangle from slightly above it. Study the figures below to get a feeling for that depth perception. We show a broken line to indicate that part of a figure that would be hidden in a physical sense.

vertical plane

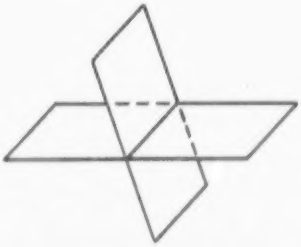
horizontal plane



a line in a plane



a line intersecting a plane



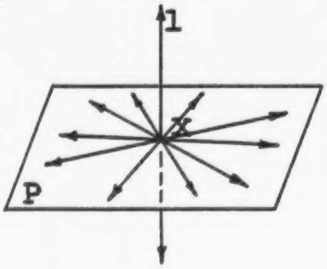
intersecting planes

nonintersecting planes

We have defined the relations, perpendicular and parallel, for two lines. We can also define these relations for a line and a plane, and for two planes.

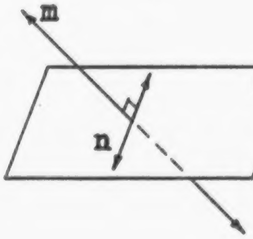
A *line is perpendicular to a plane* if and only if it is perpendicular to every line in the plane passing through the point of intersection. (The point of intersection is called the foot of the perpendicular.)

$l \perp P \longleftrightarrow l \perp$ every line in P
passing through X



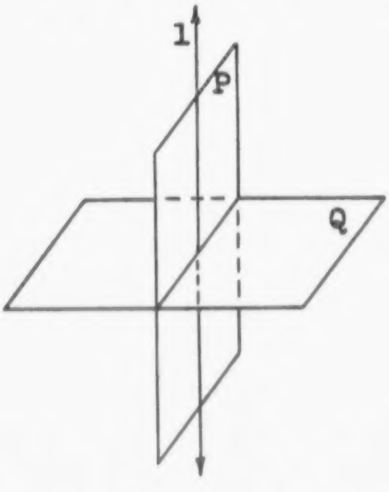
Notice that its possible for a line to be perpendicular to one line in a plane and not be perpendicular to the plane.

$m \perp n$ but $m \not\perp Q$



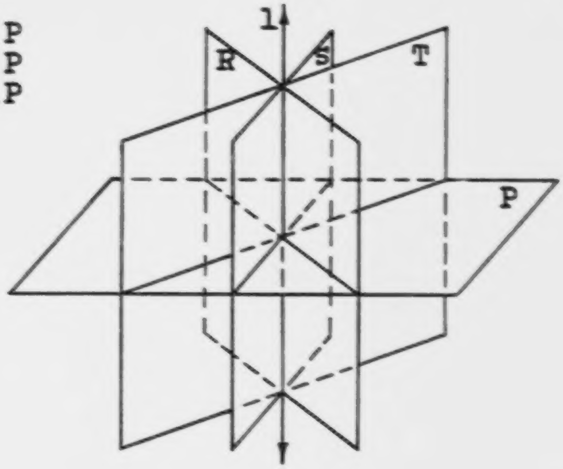
A *plane is perpendicular to a plane* if and only if it contains a line that is perpendicular to that plane.

$$\left.\begin{array}{r} P \text{ contains } l \\ l \perp Q \end{array}\right\} \longleftrightarrow P \perp Q$$



Of course, any number of planes may be perpendicular to the same plane, all of which contain the same line.

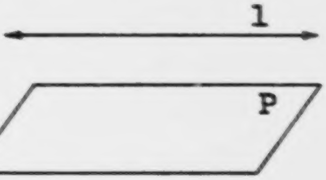
$$\left.\begin{array}{r} R \text{ contains } l \\ S \text{ contains } l \\ T \text{ contains } l \\ l \perp P \end{array}\right\} \longrightarrow \left\{\begin{array}{l} R \perp P \\ S \perp P \\ T \perp P \end{array}\right.$$



The parallel relation is simply defined for both cases.

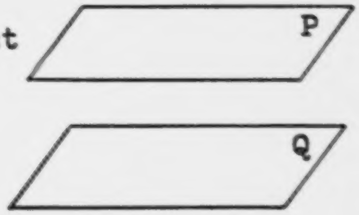
A line and a plane are parallel if and only if they do not intersect.

$l \parallel P \longleftrightarrow l$ and P do not intersect



Two planes are parallel if and only if they do not intersect.

$P \parallel Q \longleftrightarrow P$ and Q do not intersect



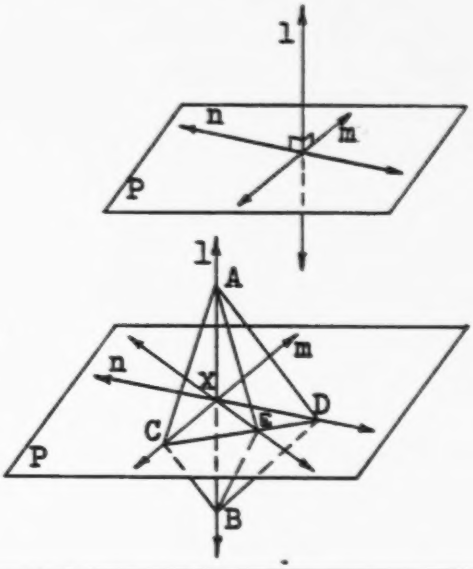
The first theorem we will state and prove will test your powers of visualizing in space. The auxiliary lines will be shown as solid so they will not be confused with hidden lines. It uses the concept of the perpendicular bisector of a segment, but remember that, in space, a segment may have any number of perpendicular bisectors.

Theorem 15-1: If a line is perpendicular to each of two intersecting lines, it is perpendicular to the plane determined by those lines.

Given: m and n determine P
$l \perp m$, $l \perp n$

Prove: $l \perp P$

Comment: If we can prove l is ⊥ to any line as a representative of all other lines in the plane through X, the definition will justify that $l \perp P$.



Argument

① Let A and B be points on l equidistant from X.
② Let C and D be points on m and n respectively.
③ Draw $\overline{CD}$
④ Draw any line in P through X intersecting $\overline{CD}$. Call the intersection E.
⑤ Draw $\overline{AC}$, $\overline{AD}$, $\overline{AE}$, $\overline{BC}$, $\overline{BD}$, $\overline{BE}$

$$\left.\begin{array}{l} \text{⑥ m is a } \perp \text{ bis. of } \overline{AB} \xrightarrow{\text{⑦}} \overline{AC} \cong \overline{BC} \\ \text{⑧ n is a } \perp \text{ bis. of } \overline{AB} \xrightarrow{\text{⑨}} \overline{AD} \cong \overline{BD} \\ \qquad\qquad\qquad\qquad\quad \text{⑩}\ \overline{CD} \cong \overline{CD} \end{array}\right\} \xrightarrow{\text{⑪}} \triangle ACD \cong \triangle BCD \xrightarrow{\text{⑫}}$$

$$\xrightarrow{\text{⑫}} \left.\begin{array}{r} \angle ACE \cong \angle BCE \\ \text{⑦}\ \overline{AC} \cong \overline{BC} \\ \text{⑬}\ \overline{CE} \cong \overline{CE} \end{array}\right\} \xrightarrow{\text{⑭}} \triangle ACE \cong \triangle BCE \xrightarrow{\text{⑮}} \left.\begin{array}{r} AE = BE \\ \text{①}\ AX = BX \end{array}\right\} \xrightarrow{\text{⑯}}$$

$$\xrightarrow{\text{⑯}} \overleftrightarrow{EX} \text{ is a } \perp \text{ bis. of } \overline{AB} \quad (\overleftrightarrow{EX} \perp l)$$

Since we have shown that $\overleftrightarrow{EX}$ is any line in P and ⊥ l through X, then l is ⊥ to all lines in P through X and $l \perp P$.

The following are existence and uniqueness theorems very similar to those involving points and lines in a plane. We will state them here and ask you to illustrate them with a figure in the exercises.

A: At a point on a line, there is exactly one plane perpendicular to the line.

B: Through a point not on a line, there is exactly one plane perpendicular to the line.

C: At a point on a plane, there is exactly one line perpendicular to the plane.

D: Through a point not on a plane, there is exactly one line perpendicular to the plane.

The last statement allows us to give another definition for distance.

The _distance from a point to a plane_ is the length of the perpendicular segment from the point to the plane.

E: If a plane is perpendicular to a line, it contains every line that is perpendicular to the given line at the point of intersection.

EXERCISE SET 15.1

Draw figures representing each of the following.

1. A vertical plane and a line perpendicular to it.
2. A vertical plane and a line perpendicular to one line in the plane but not perpendicular to the plane.
3. Two parallel planes that are neither horizontal nor vertical.
4. A plane perpendicular to each of two parallel planes.
5. Three planes whose common intersection is a single line.
6. Three planes whose common intersection is a single point.
7. Three planes, each of which intersects the other two so that the three lines of intersection are three distinct parallel lines.

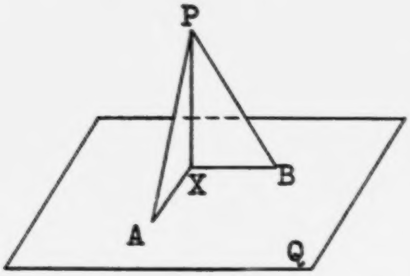
Complete

8. A line and a plane are perpendicular if and only if ___
9. Two planes are perpendicular if and only if ___
10. A line and a plane are parallel if and only if ___
11. Two planes are parallel if and only if ___

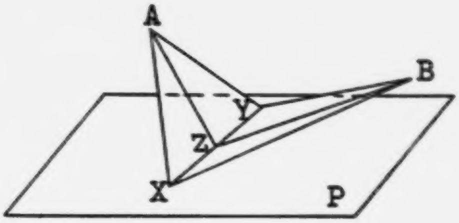
12. Write the reasons for statements ⑦, ⑨, and ⑯ in the proof of theorem 15-1.

13. Illustrate the existence and uniqueness statements A-E with a figure for each and the appropriate implication.

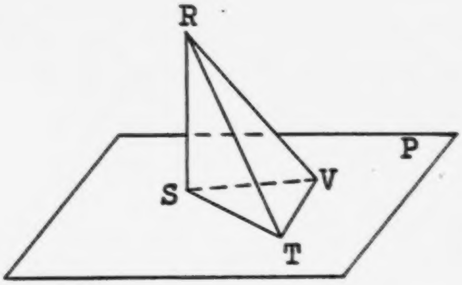
14. Given: $\overline{PX} \perp Q$, $\overline{PA} \cong \overline{PB}$
Prove: $\overline{AX} \cong \overline{BX}$



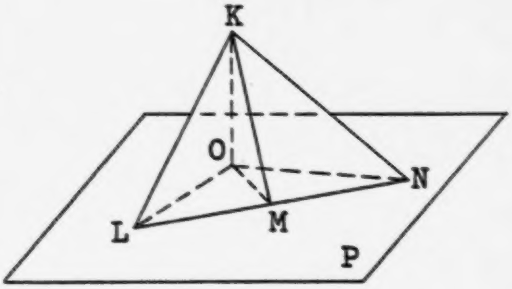
15. Given: $\overline{AX} \cong \overline{BX}$, $\overline{AY} \cong \overline{BY}$
Prove: $\overline{AZ} \cong \overline{BZ}$



16. Given: $\overline{RS} \perp P$, $\angle STV \cong \angle SVT$
Prove: $\angle RTV \cong \angle RVT$



17. Given: $\overline{KO} \perp \overline{OL}$, $\overline{KO} \perp \overline{OM}$
$\overline{OL} \cong \overline{ON}$, $\overline{OM}$ is a median of ΔLON
Prove: $\overline{KM} \perp \overline{LN}$

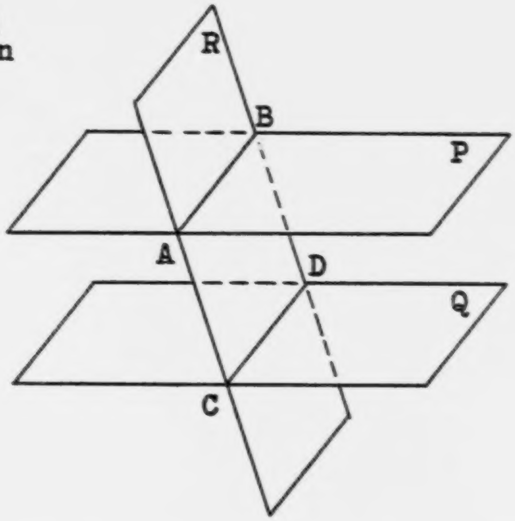


The next set of theorems involve parallel planes and lines in space. Once again, we will prove the first one and ask you to illustrate the remainder with a figure. It is often helpful to construct a physical model for relations in space.

Theorem 15-2: If a plane intersects each of two parallel planes, the lines of intersection are parallel.

Given: $P \parallel Q$
R intersects P and Q
in $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ as shown

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$



Argument

$$\left.\begin{array}{r} \text{①}\ \overleftrightarrow{AB} \text{ lies in } P \\ \text{②}\ \overleftrightarrow{CD} \text{ lies in } Q \\ \text{③}\ P \parallel Q \end{array}\right\} \xrightarrow{\text{④}} \left.\begin{array}{r} \overleftrightarrow{AB} \text{ and } \overleftrightarrow{CD} \text{ do not intersect} \\ \text{⑤}\ \overleftrightarrow{AB} \text{ and } \overleftrightarrow{CD} \text{ are both in } R \end{array}\right\} \xrightarrow{\text{⑥}} \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$$

$$\therefore \left.\begin{array}{r} R \text{ intersects } P \text{ and } Q \\ P \parallel Q \end{array}\right\} \longrightarrow \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$$

F: If a line is perpendicular to one of two parallel planes, it is perpendicular to the other.

G: If two planes are perpendicular to the same line, they are parallel.

H: If two planes are parallel to a third plane, they are parallel to each other.

I: If two lines are perpendicular to the same plane, they are parallel.

J: If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other.

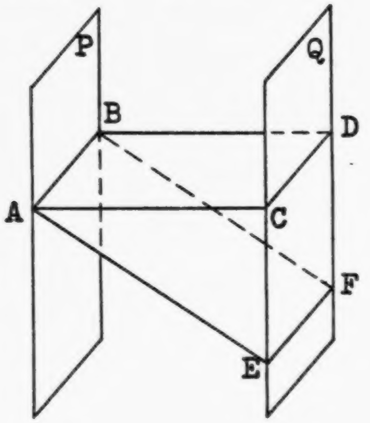
K: If two lines are parallel to a third line, they are parallel to each other.

L: Parallel planes are everywhere equidistant.

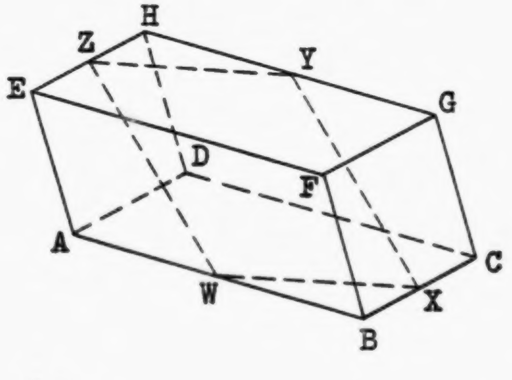
EXERCISE SET 15.2

1. Write the reason for statement ⑥ in the proof of theorem 15-2.
2. Illustrate statements F-L with a figure for each and the appropriate implication.
3. Two lines parallel to the same plane (must, may, cannot) be parallel to each other.
4. Two planes perpendicular to the same line (must, may, cannot) be perpendicular to each other.
5. Two planes perpendicular to the same plane (must, may, cannot) be perpendicular to each other.
6. If a line is parallel to one of two parallel planes and does not lie in either of them, then it (must, may, cannot) be parallel to the other.
7. Two planes parallel to the same line (must, may, cannot) be parallel to each other.
8. If a line is perpendicular to one of two perpendicular planes and does not lie in the other, then it (must, may, cannot) be parallel to the other.
9. If a plane contains one of two perpendicular lines, then it (must, may, cannot) be perpendicular to the other.
10. Two lines parallel to the same plane (must, may, cannot) be perpendicular to each other.

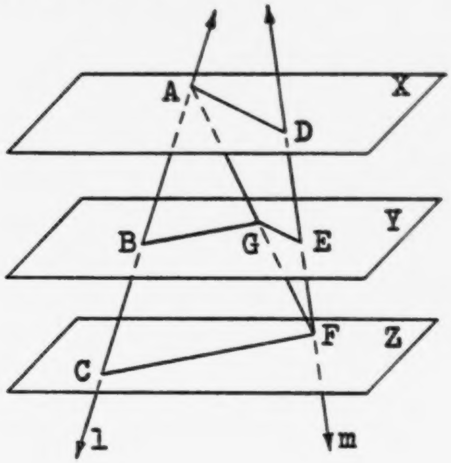
11. Given: $P \parallel Q$
    Prove: $\overline{CD} \parallel \overline{EF}$



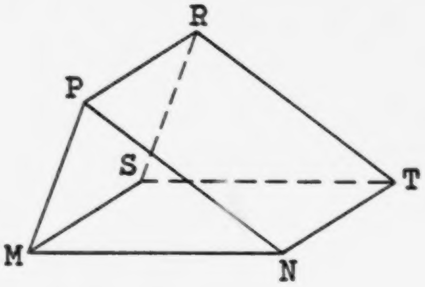
12. Given: $ABCD \parallel EFGH$
    $\overline{AB} \cong \overline{HG}$, $\overline{BC} \cong \overline{EH}$
    $\angle ABC \cong \angle GHE$
    W, X, Y, and Z are midpoints
    Prove: WXYZ is a ▱



13. Given: l and m are skew lines
X ∥ Y ∥ Z, $\overline{AB} \cong \overline{BC}$
$\overline{AF}$ intersects Y at G
Prove: $\overline{DE} \cong \overline{EF}$

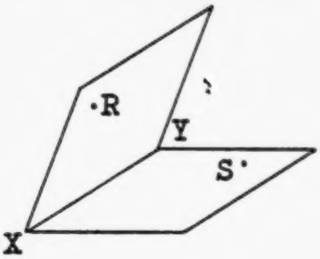


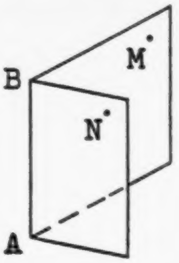
14. Given: MNTS is a ▱
MPRS is a ▱
Prove: ∠MNP ≅ ∠STR



Dihedral Angles

A *dihedral angle* is the union of two intersecting noncoplanar half-planes and their common edge.



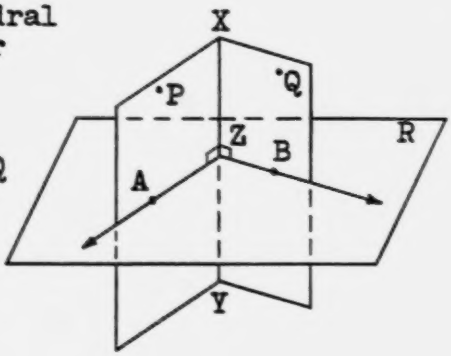


The common edge is the *edge* of the dihedral angle and a half-plane with the edge is a *face* of the dihedral angle.

A dihedral angle is named by naming a point in each face separated from two points in the edge by hyphens. The dihedral angles shown are R-XY-S and M-AB-N.

The intersection of a dihedral angle and a plane perpendicular to its edge is a plane angle of a dihedral angle.

AZB is a plane angle of P-XY-Q



It can be proved that all plane angles of a dihedral angle are congruent. This allows the following definition.

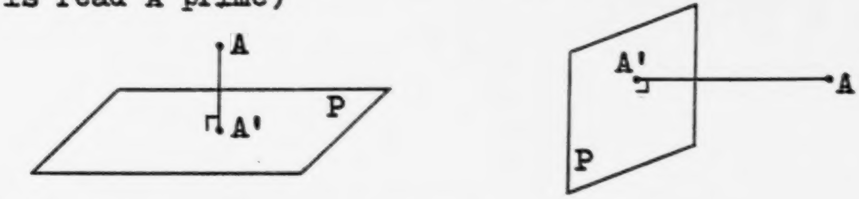
The measure of a dihedral angle is the measure of its plane angle.

We can obviously define acute, right, and obtuse dihedral angles to be consistent with the definitions for plane angles. The same is true for the interior and exterior of a dihedral angle.

Projections

The word projection is descriptive enough to suggest its meaning in geometry. An important idea to keep in mind in connection with projections is the concept of perpendicular.

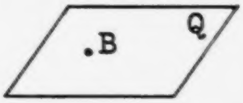
The projection of a point into a plane is the foot of the perpendicular from the point to the plane. (A' is read A prime)



A' is the projection of A into P.

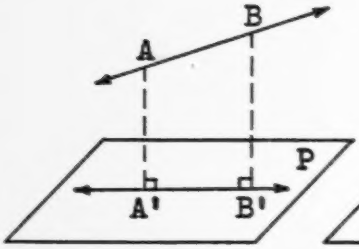
If a point lies in a plane, it is defined to be its own projection into the plane.

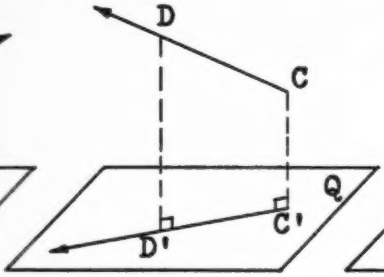
B is the projection of B into Q.

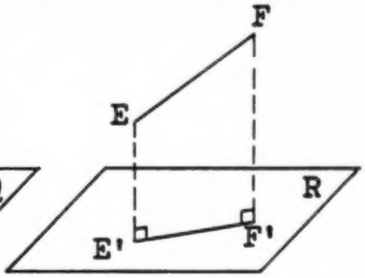


The projection of any set of points into a plane is the set of all points in the plane which are projections of the points of the given set.

In general, the projection of a line, ray, or segment into a plane is a line, ray, or segment respectively. We will use the abbreviation pn for "is the projection of".

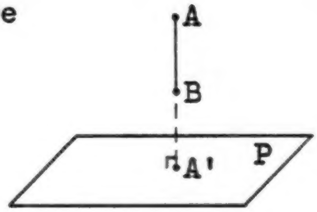






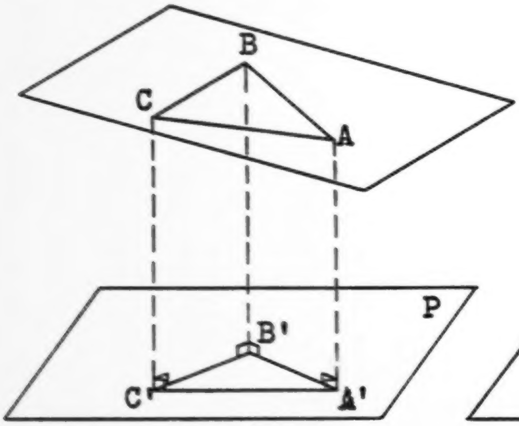
$\overleftrightarrow{A'B'}$ pn $\overleftrightarrow{AB}$ in P $\overrightarrow{C'D'}$ pn $\overrightarrow{CD}$ in Q $\overline{E'F'}$ pn $\overline{EF}$ in R

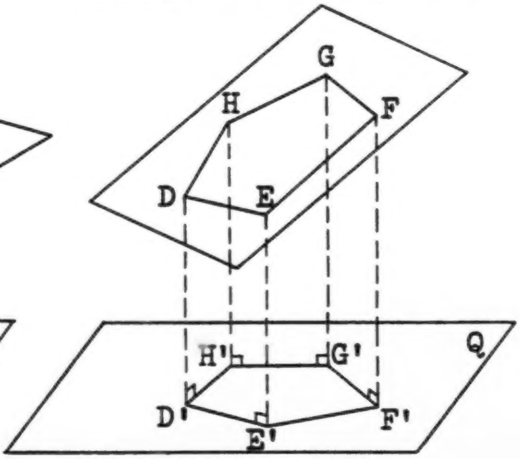
The exception to this rule would be the case in which the line, ray, or segment lies in a line perpendicular to the plane. All points of $\overline{AB}$ have the same projection which is a single point. A' and B' are the same point.



A' pn AB in P

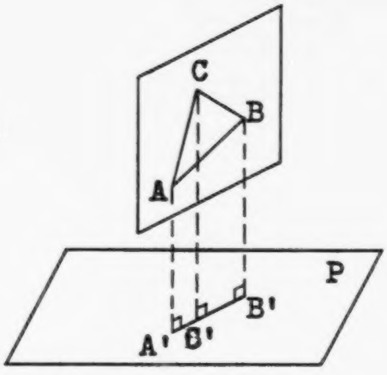
The same generalization, with a similar exception, can be made for projections of polygons into a plane.





$\Delta A'B'C'$ pn ΔABC in P D'E'F'G'H' pn DEFGH in Q

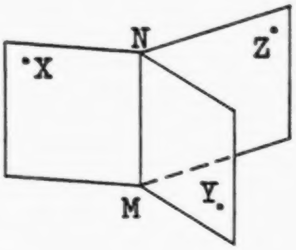
The exception would be the case in which the plane of the given polygon is perpendicular to the plane it is being projected into. The projection would then be a segment.



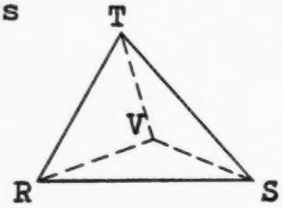
A'B' pn ABC in P

EXERCISE SET 15.3

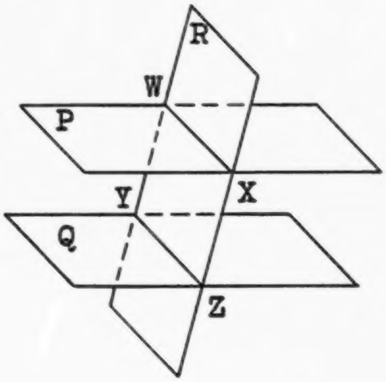
1. Name all the dihedral angles in the figure.



2. Define a right dihedral angle.
3. Draw an obtuse dihedral angle.
4. Name six dihedral angles in the figure.

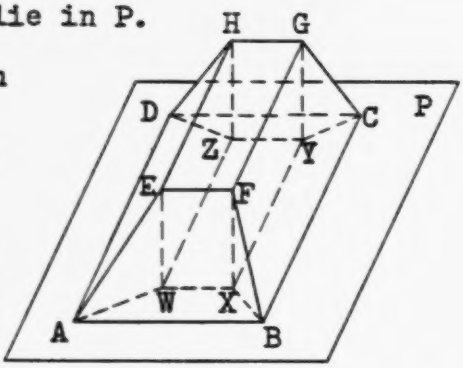


5. How many dihedral angles are formed by the walls, ceiling, and floor of your room?
6. Prove: If two parallel planes are intersected by a third plane, the alternate interior dihedral angles are congruent. (Hint: Introduce an auxiliary plane.)

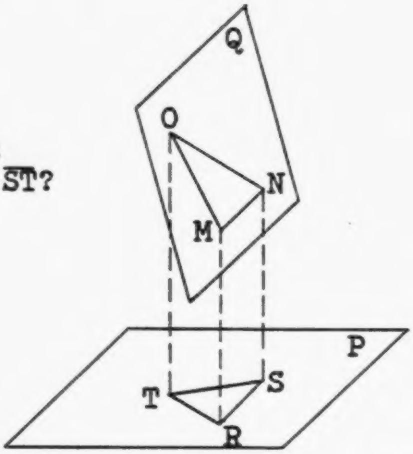


7. A, B, C, D, W, X, Y, Z all lie in P.
$\overline{EW}$, $\overline{FX}$, $\overline{GY}$, $\overline{HZ}$ all $\perp$ P.
Name the projections of each of the following in P.

a) E
b) $\overline{FG}$
c) $\overline{HD}$
d) EFGH
e) $\overline{GY}$
f) B
g) ABFE
h) HZYG
i) $\triangle$AWE
j) $\overline{AD}$
k) $\angle$GFB
l) DZYC

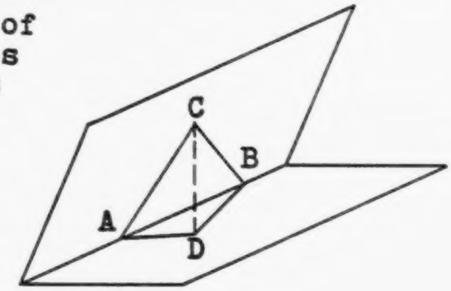


8. $\triangle$MNO lies in Q, Q $\nparallel$ P
$\triangle$RST pn $\triangle$MNO in P, $\overline{MN} \parallel$ P
MN = 5, MO = 6, NO = 7
What can you conclude about the lengths of $\overline{RS}$, $\overline{RT}$, and $\overline{ST}$?



9. Can the length of the projection of a segment ever be greater than the length of the segment?

10. Given: $\overline{AB}$ lies in the edge of a dihedral $\angle$. C lies in one face and D is the projection of C in the other face. $\triangle$ABC is equilateral
Prove: $\triangle$ABD is isosceles but not equilateral

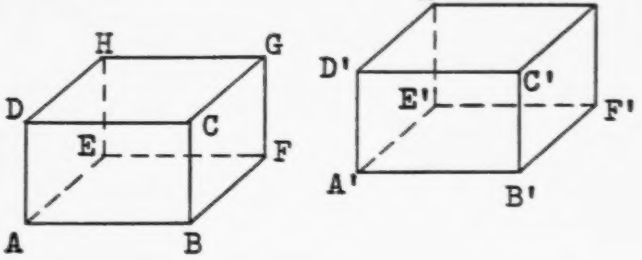


CHAPTER 16 - TRANSFORMATIONS
or WHERE DO I GO FROM HERE

Mappings and Reflection

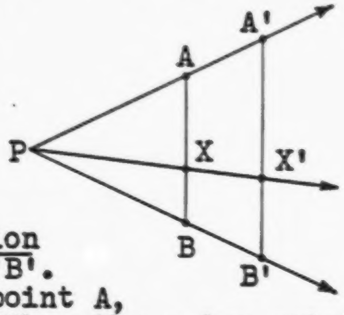
One of the more recent developments in the high school math curriculum is an emphasis on the ideas of transformations. This will give you a fresh approach to some new concepts in geometry as well as a fresh viewpoint on some of the concepts already developed. Furthermore, the foundations of transformations considered here will be basic groundwork for some very important mathematics that you may study later.

As you recall, we have defined space as the set of all points. In this abstract sense, every point must be considered as fixed and can never be moved. We may move a box in a physical sense from one position to another, but in the abstract sense we would think of associating the corners of the box with a new set of points in space. This allows us to have a feeling for "motion" without actually moving the points themselves.



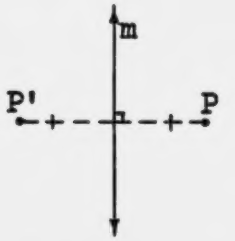
We will limit our discussion here to a plane and we will regard a transformation as a rule which associates a given set of points in the plane with another set of points in the plane.

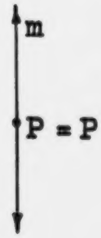
Let $\overline{AB}$ be the first set and $\overline{A'B'}$ the second set. If the rule is to draw rays with endpoint P intersecting both segments, we say that A is "mapped" onto A', B is mapped onto B'. In fact, any point X, of $\overline{AB}$, is mapped onto some point X', of $\overline{A'B'}$, by this rule. The transformation is considered a mapping of $\overline{AB}$ onto $\overline{A'B'}$. The point A' is called the image of point A, and A is called the preimage of A'. When the prime notation (X') is used, X will always denote the preimage, or first point, and X' will denote the image of X, or second point.



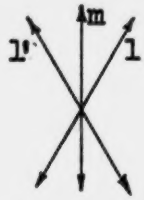
One of the most fundamental transformations in geometry is the reflection.

If P is not on line m, the reflection image of P over line m is the point P' if and only if m is the perpendicular bisector of $\overline{PP'}$. If P is on m, the reflection image of P is P itself. In the second case, P is called a fixed point.

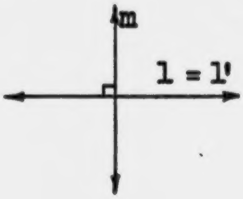




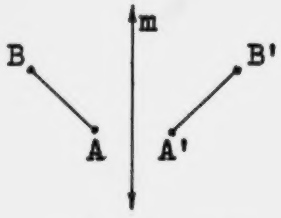
The line m is called the mirror (or reflecting line) of the reflection. The reflection will be denoted by M, and if the mirror, such as m, is specified, the reflection in the line m will be written M_m. If P' is the image of P in m, we write $M_m(P) = P'$. For the fixed point case, we would write $M_m(P) = P$. Some other reflections with the appropriate notation are shown below.



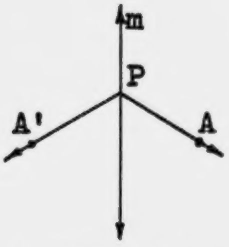
$M_m(l) = l'$



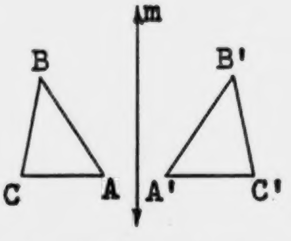
$M_m(l) = l$



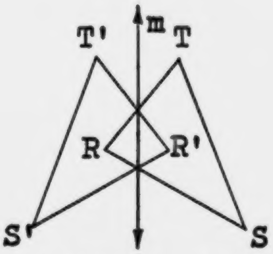
$M_m(\overline{AB}) = \overline{A'B'}$



$M_m(\overrightarrow{PA}) = \overrightarrow{PA'}$



$M_m(\Delta ABC) = \Delta A'B'C'$



$M_m(\Delta RST) = \Delta R'S'T'$

As we study each transformation, we will be particularly interested in the properties of the figures which remain unchanged, or are preserved under the transformation. We say such properties are invariant. The preceding figures suggest the following as postulates for reflection transformations.

Postulate 14: The reflection image of a line is a line.

Postulate 15: A reflection transformation preserves the betweenness property of points.

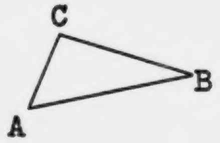
Postulate 16: A segment and the reflection image of that segment have the same length. (reflection preserves distance between points)

Postulate 17: The reflection image of an angle is an angle of the same measure.

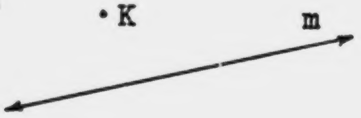
Postulate 18: The reflection image of a convex polygon reverses the orientation of the vertices. If the vertices of polygon ABCD... are in a clockwise orientation, then the vertices of its image A'B'C'D'... are in a counter-clockwise orientation.

EXERCISE SET 16.1

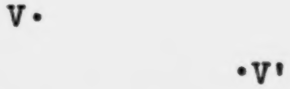
1. If you move this book 2" to the right, are the vertices of ΔABC, as abstract points, also moved 2" to the right? Are the vertices of the triangle in your friends book the same points as those in your book? Why or why not?



2. If $M_m(F) = F'$, name the image. Name the preimage.
3. If D' is the reflection image of D in line l, how is line l related to $\overline{DD'}$?
4. K is the preimage of point K' in line m. Copy the figure and construct point K'.

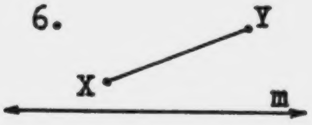


5. V' is the image of V in line r. Copy the figure and construct r.



In each figure below, m is the mirror of a reflection transformation. Copy the figure and draw the image of the given set of points in m and label it appropriately.

6.



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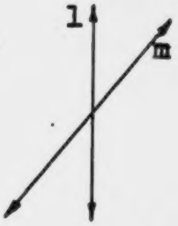

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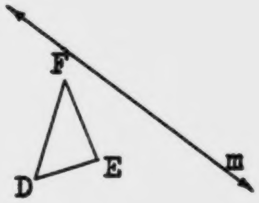

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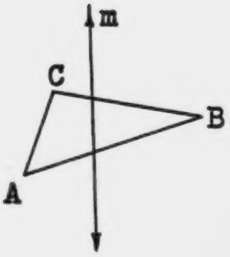

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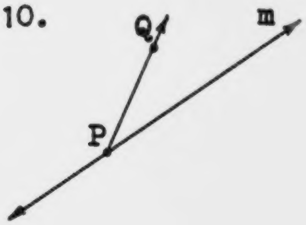

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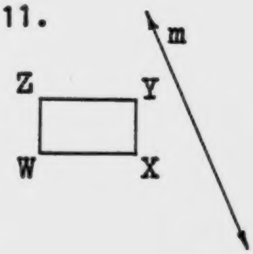

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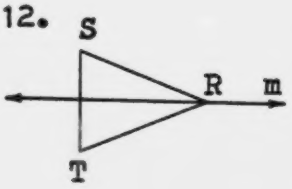

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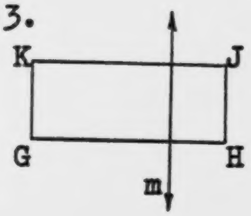
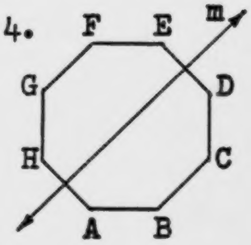


14.



15. If a property of a set of points does not change under a reflection transformation, that property is said to be ______.

Use postulates 14-18 to complete the following.

16. $M_m(\overline{AB}) = \overline{A'B'} \longrightarrow$ ______.

17. $M_m(\Delta ABC) = \Delta A'B'C'$. If the vertices of ΔABC are oriented counterclockwise, then the vertices of $\Delta A'B'C'$ are oriented ______.

18. $M_m(\overline{XY}) = \overline{X'Y'}$. If P is a point between X and Y, ______.

19. If $M_m(A) = A'$ and $M_m(B) = B'$, then $M_m(\overleftrightarrow{AB}) =$ ______.

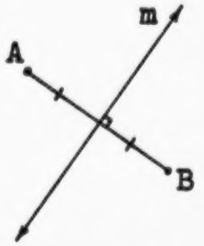
20. $M_m(\angle RST) = \angle R'S'T' \longrightarrow$ ______.

Symmetry

The idea of symmetry is a very important one in geometry and is closely related to the reflection transformation. The definition of symmetry with respect to a line follows.

Two points are symmetric with respect to a line if and only if the line is the perpendicular bisector of the segment joining the two points.

A figure is symmetric with respect to a line if and only if there is a point of symmetry for every point of the figure.

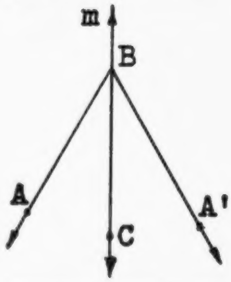


A is symmetric to B with respect to m if and only if m is the perpendicular bisector of $\overline{AB}$. Doesn't this also define the reflection image of A in m as B? Although symmetry is a property of a set of points, and reflection is a mapping of a set of points onto a set of points, when we have one we have the other. We can make good use of this equivalence.

The above figure could also be used to illustrate the symmetry of a segment with its unique perpendicular bisector as the symmetry line. In this case, every point on the segment has a symmetry point on the segment and we would say $\overline{AB}$ is symmetric to $\overline{BA}$ with respect to m.

Another important symmetry involves an angle and its bisector. An angle bisector is actually a ray but in this discussion we will refer to it as the line containing that ray.

Theorem 16-1: The bisector of an angle is a symmetry line for the angle.



Let $\overrightarrow{BA'}$ be the reflection image of $\overrightarrow{BA}$ in m
Then $M_m(\angle ABC) = \angle A'BC$
$m\angle ABC = m\angle A'BC$ since $\angle$ measure is preserved
Thus m is the $\angle$ bisector of $\angle ABA'$

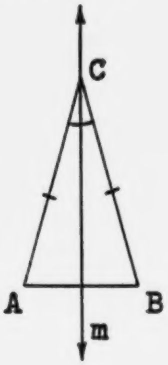
$\therefore$ The angle bisector is a symmetry line for the angle.

The above argument is written in an informal paragraph form. Throughout this chapter we will use paragraph proof where appropriate, flow proof where appropriate, or we may combine the two where appropriate.

We will accept the following statement of symmetry for polygons.

A polygon is symmetric with respect to a line if and only if every vertex of the polygon is symmetric to another, vertex of the polygon with respect to that line.

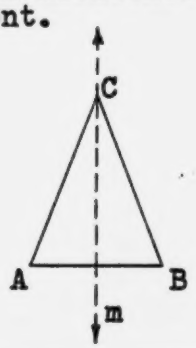
Theorem 16-2: The angle bisector of the vertex angle of an isosceles triangle is a symmetry line for the triangle.



In the isosceles ΔABC shown, let m be the angle bisector of the vertex $\angle C$.
By thm 16-1, the image of $\overrightarrow{CA}$ in m is $\overrightarrow{CB}$.
Then the image of A must lie on $\overrightarrow{CB}$.
Since distances are preserved in reflection, the image of A must be the same distance from C that A is from C.
Since $AC = BC$, the image of A is B and the image of B is A.
Thus $M_m(\Delta ABC) = \Delta BAC$
$\therefore$ The angle bisector, m, is a symmetry line for the isosceles ΔABC.

With these theorems and definitions, we can use a different and interesting approach to the proofs of some of the theorems we have already proved.

Theorem 16-3 (11-5): The base angles of an isosceles triangle are congruent.



Given: Isos. ΔABC with $\overline{AC} \cong \overline{BC}$

Prove: $\angle A \cong \angle B$

Argument

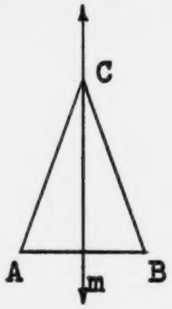
Draw the angle bisector of $\angle C$. Call it m.
Reflect ΔABC in m. $M_m(\Delta ABC) = \Delta BAC$ by thm 16-2.
Since the measures of angles are preserved in reflection,
$m\angle BAC = m\angle ABC$
$\therefore \angle A \cong \angle B$

The next theorem was an exercise in chapter 11.

Theorem 16-4: The bisector of the vertex angle of an isosceles triangle is the same line as the perpendicular bisector of the base.

Given: Isosceles $\triangle ABC$
m is the $\angle$ bisector of vertex $\angle C$

Prove: m is the $\perp$ bisector of $\overline{AB}$

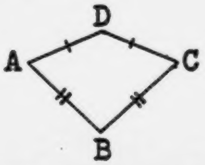


Argument

m is a symmetry line for isos. $\triangle ABC$ by thm 16-2.
So A is symmetric to B with respect to m.
$\therefore$ m is the $\perp$ bisector of $\overline{AB}$ by def. of line symmetry.

Definition: A kite is a quadrilateral with two distinct pairs of adjacent sides congruent.

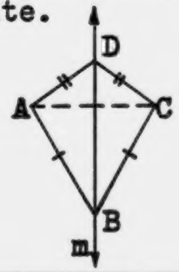
$$\left.\begin{array}{l}\overline{AB}\cong\overline{CB}\\ \overline{AD}\cong\overline{CD}\end{array}\right\}\longleftrightarrow \text{ABCD is a kite}$$



Theorem 16-5: The line determined by the intersections of each pair of congruent sides of a kite is a symmetry line for the kite.

Given: ABCD is a kite
with $\overline{AB}\cong\overline{CB}$, $\overline{AD}\cong\overline{CD}$

Prove: $\overleftrightarrow{BD}$ is a symmetry line for the kite



Argument

Draw $\overline{AC}$
Let m be the $\perp$ bisector of $\overline{AC}$.
Since $\triangle ACD$ is isos., m is also the $\angle$ bisector of vertex $\angle D$.
Since $\triangle ACB$ is isos., m is also the $\angle$ bisector of vertex $\angle B$.
Thus m contains D and m contains B and m is the line $\overleftrightarrow{BD}$.
A and C are reflection images of each other in $\overleftrightarrow{BD}$.
B and D are their own images in $\overleftrightarrow{BD}$.
$\therefore$ $\overleftrightarrow{BD}$ is a symmetry line for the kite.

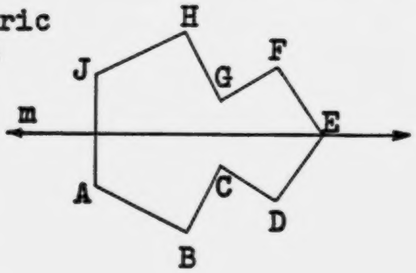
Remember this theorem as we will use it shortly.

EXERCISE SET 16.2

1. How many symmetry lines are there for each of the following plane figures?
   a) An isosceles triangle that is not equilateral
   b) A rhombus that is not a square
   c) A parallelogram that is not a rhombus
   d) An angle
   e) A segment
   f) A line
   g) A scalene triangle
   h) A square
   i) An equilateral triangle
   j) An isosceles trapezoid
   k) A regular pentagon
   l) A regular hexagon
   m) A circle
   n) A kite

2. Draw a square and all of its symmetry lines.

3. The figure shown is symmetric with respect to m. Make a statement expressing this symmetry as a reflection transformation.



4. If $\triangle RST$ is isosceles and $\overleftrightarrow{XY}$ is a symmetry line for $\triangle RST$, what else can you say about $\overleftrightarrow{XY}$?

5. If DEFG is a kite with $\overline{DE} \cong \overline{DG}$ and $\overline{EF} \cong \overline{GF}$, name a symmetry line for the kite.

Rotation

If two reflecting lines intersect in a plane, we may consider a composite transformation of two reflections of point A as shown.

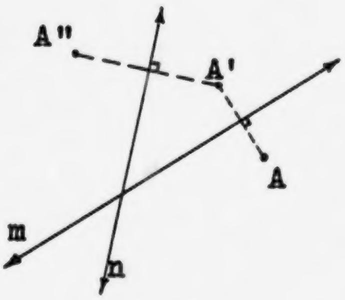


Fig. 1

$M_m(A) = A'$ and $M_n(A') = A''$ (A double prime). We call A" the product of M_m and M_n. We say that M_n is applied to the "previously determined" transformation M_m, so we write the product

$$M_n \cdot M_m(A) = A''$$

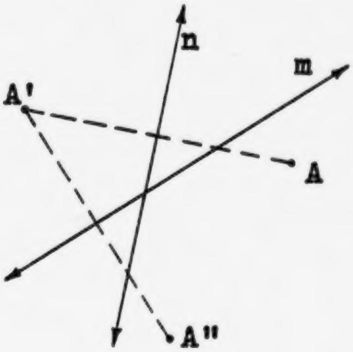
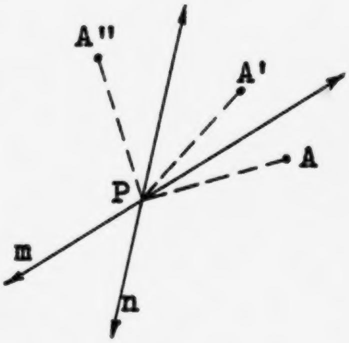


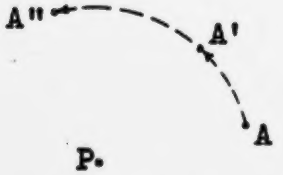
Fig. 2

The order is important as this "product" is not commutative. $M_m \cdot M_n(A) = A''$ would mean that reflection M_n is performed first and M_m applied to the result. Fig. 2 shows that A" would not be the same point in both products.



Let's redraw Fig. 1 showing the segments from the point of intersection, P, to A, A', and A". Since m is the perpendicular bisector of AA', then PA = PA'. Since n is the perpendicular bisector of A'A", then PA' = PA".
Therefore, PA = PA' = PA".

This means that A" can be considered a rotation of A about point P as center.



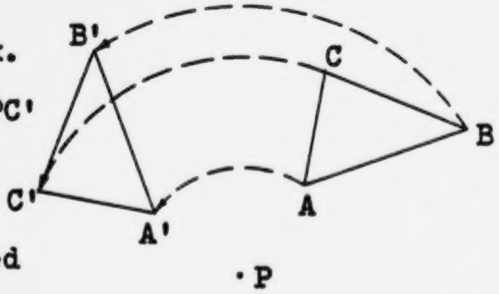
A transformation R_p is a rotation if and only if $R_p = M_m \cdot M_n$ where m and n intersect at P.

For the previous example, we would write
$R_p(A) = M_n \cdot M_m(A) = A''$.

The rotation, or turning, may be considered without the reflecting lines as the rotation of a figure in a plane about a given point in the plane as center.

$R_p(\Delta ABC) = \Delta A'B'C'$

Each point is rotated through the same angle with P as vertex. If the angles were drawn, we would have $m\angle APA' = m\angle BPB' = m\angle CPC'$ Since it is the product of two reflections, the orientation of the vertices (ABC counterclockwise) is reversed twice and therefore is preserved in the image.

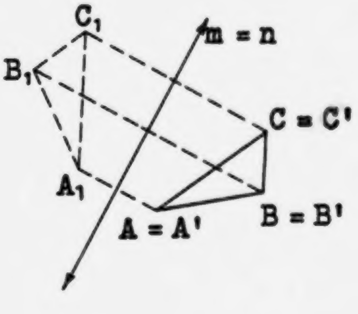


By defining the rotation as a product of reflections, we know that all properties that are invariant under reflection will also be invariant under rotation.

An identity rotation is a rotation in which the reflecting lines, m and n, are the same line. That is, they intersect in all their points.

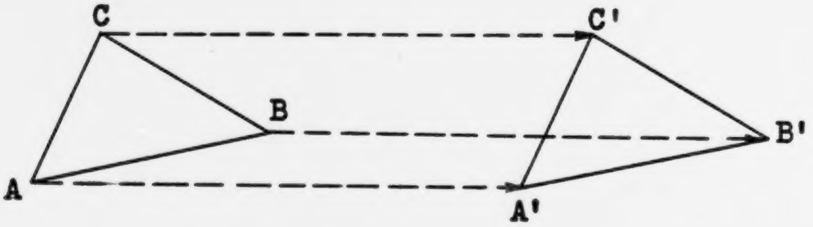
If ΔABC is reflected in m, and its image, $\Delta A_1B_1C_1$, reflected in n, we have $R(\Delta ABC) = \Delta A'B'C' = \Delta ABC$

That is, the triangle may be rotated onto itself. There are special cases in which the idea of this transformation is useful.



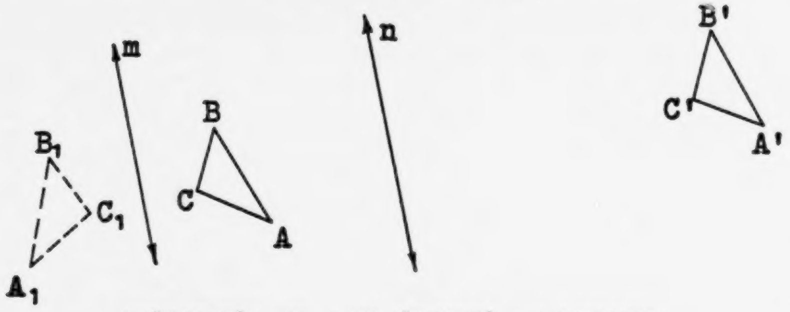
Translation

Another transformation may be thought of as a "slide" motion. If every point of ABC is mapped onto a point exactly the same distance to the right, the transformation is called a translation.



The "slide" may be in any direction but does not involve any rotation. We write $T(\Delta ABC) = \Delta A'B'C'$.

Just as a rotation is defined as a product of two reflections, so may a translation be defined. In this case, the two reflecting lines are parallel.



$$T(\Delta ABC) = M_n \circ M_m(\Delta ABC) = \Delta A'B'C'$$

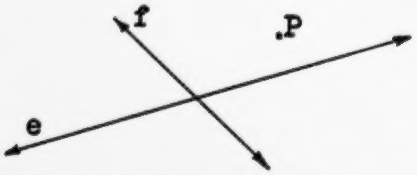
A transformation is a translation if and only if $T = M_m \circ M_n$, where $m \parallel n$ or $m = n$.

The second case, $m = n$, would result in an identity translation. This allows a figure to be translated onto itself.

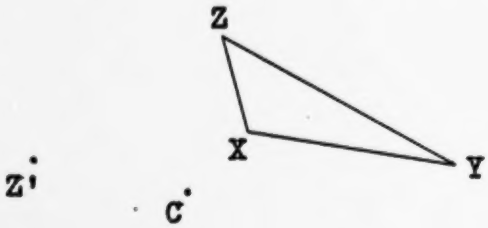
The postulates for both the rotation and translation transformations would be the same as for reflection except for postulate 18, in which the orientation of the vertices of a convex polygon is preserved instead of reversed.

EXERCISE SET 16.3

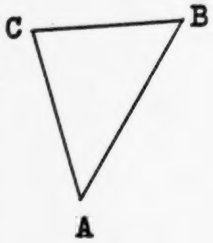
1. Copy the figure.
   a) Construct P_1 such that $M_e \cdot M_f(P) = P_1$.
   b) Construct P_2 such that $M_f \cdot M_e(P) = P_2$.
   c) With the intersection of e and f as center, draw a circle through P. Does the circle contain P_1 and P_2?



2. Copy the figure and construct $\Delta X'Y'Z'$ such that $R_C(\Delta XYZ) = \Delta X'Y'Z'$. (Hint: Draw $\angle ZCZ'$ and construct two other angles congruent to this angle.)

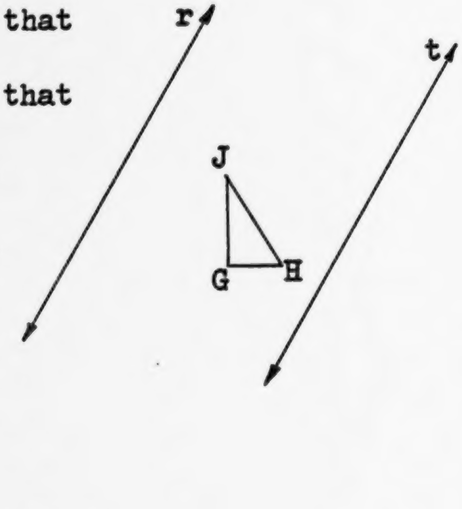


3. Copy the figure and construct $\Delta A'B'C'$ such that $R_A(\Delta ABC) = \Delta A'B'C'$ with the angle of rotation 60° clockwise.

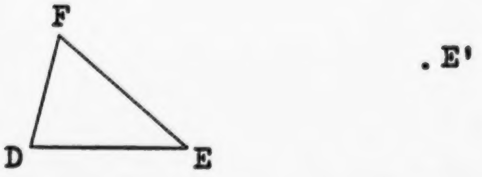


4. If $R_D(\Delta DEF) = \Delta DEF$, then R is an ________ rotation.

5. Copy the figure in which $r \parallel t$.
   a) Construct $\Delta G_1H_1J_1$ such that $M_r \cdot M_t(\Delta GHJ) = \Delta G_1H_1J_1$.
   b) Construct $\Delta G_2H_2J_2$ such that $M_t \cdot M_r(\Delta GHJ) = \Delta G_2H_2J_2$.
   c) Draw $\overline{GG_1}$, $\overline{HH_1}$, and $\overline{JJ_1}$. Is $GG_1 = HH_1 = JJ_1$?
   d) Draw $\overline{GG_2}$, $\overline{HH_2}$, and $\overline{JJ_2}$. Is $GG_2 = HH_2 = JJ_2$?



6. Copy the figure and draw $\triangle D'E'F'$ such that $T(\triangle DEF) = \triangle D'E'F'$.



7. State whether or not each of the following is invariant under a rotation transformation.

   a) segment length
   b) angle measure
   c) betweenness of points
   d) orientation of vertices of a polygon

8. Repeat exercise 7 for a translation transformation.

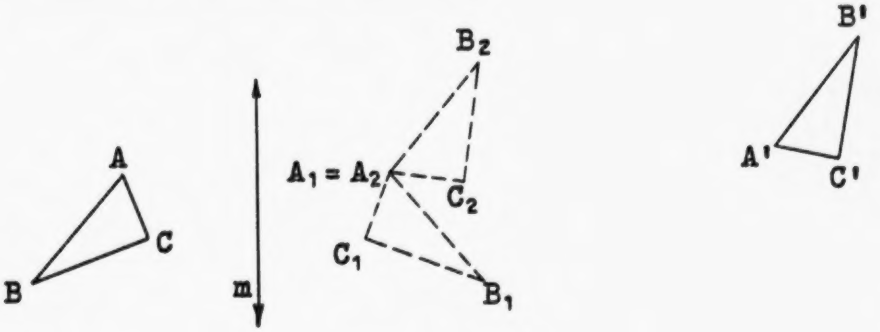
Rigid Transformations or Isometries

Each of the three transformations, reflection, rotation, and translation, is called a rigid transformation or isometry.

An isometry of the plane is a transformation such that distances between points and measures of angles are invariant under the transformation.

Any product of the three transformations is also an isometry since each of the three preserves these same properties.

Let $H(\triangle ABC) = T \circ R_{A_1} \circ M_m(\triangle ABC)$



Then $H(\triangle ABC) = \triangle A'B'C'$ is an isometry

Do you see any relationship between isometries and our previous concept of congruence? Doesn't an isometry, or rigid transformation, require that the image of any figure be the same size and shape as the figure? This will allow us to give an alternate definition for congruence.

Two figures, I and II, are congruent if and only if there is an isometry such that the image of I is II.

With this definition we can actually prove the triangle congruence statements that were accepted as postulates earlier. There would be more than one possible proof for each theorem and the strategy would depend upon the orientation of the given figures. For example, more transformations would be required for the randomly placed triangles in Figure 1 than for those in Figure 2 and Figure 3.

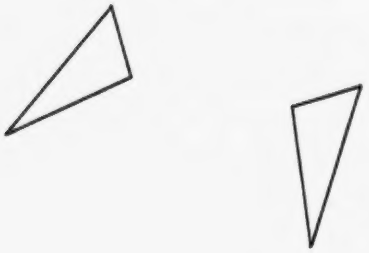


Fig. 1

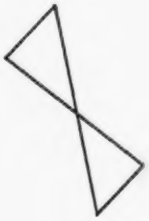


Fig. 2

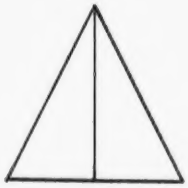


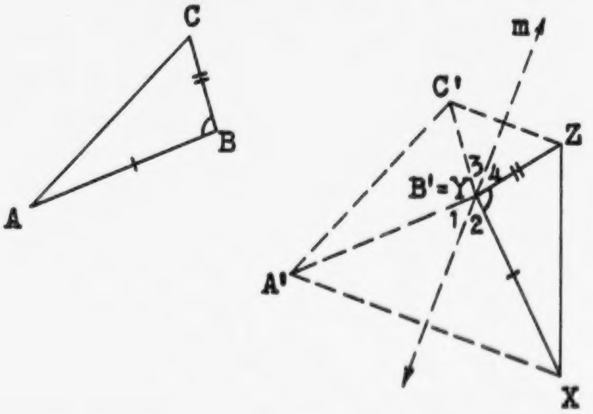
Fig. 3

We will show one possibility for each of the SAS and SSS correspondences.

Theorem 16-6: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the triangles are congruent.

Given: $\overline{AB} \cong \overline{XY}$
$\angle B \cong \angle Y$
$\overline{BC} \cong \overline{YZ}$

Prove: $\triangle ABC \cong \triangle XYZ$



Argument

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① Translate $\triangle ABC$ so that B maps onto Y. $T(\triangle ABC) = \triangle A'YC'$.
② Let m be the bisector of $\angle A'YX$. $\angle 1 \cong \angle 2$

$$\left.\begin{array}{r} \left.\begin{array}{l} \text{③}\ \angle ABC \cong \angle A'YC' \\ \text{④}\ \angle ABC \cong \angle XYZ \end{array}\right\} \xrightarrow{\text{⑤}} \angle A'YC' \cong \angle XYZ \\ \text{②}\ \angle 1 \cong \angle 2 \end{array}\right\} \to \cdots \xrightarrow{\text{⑥}} \angle 3 \cong \angle 4 \xrightarrow{\text{⑦}}$$

$$\xrightarrow{\text{⑦}} m \text{ bisects } \angle C'YZ$$

$$\left.\begin{array}{r} \left.\begin{array}{r} \left.\begin{array}{l} \text{⑧}\ \overline{AB} \cong \overline{A'Y} \\ \text{⑨}\ \overline{AB} \cong \overline{XY} \end{array}\right\} \xrightarrow{\text{⑩}} \overline{A'Y} \cong \overline{XY} \xrightarrow{\text{⑪}} \triangle A'XY \text{ is isos.} \\ \text{②}\ m \text{ bisects } \angle A'YX \end{array}\right\} \xrightarrow{\text{⑫}} M_m(A') = X \\ \left.\begin{array}{r} \left.\begin{array}{l} \text{⑬}\ \overline{CB} \cong \overline{C'Y} \\ \text{⑭}\ \overline{CB} \cong \overline{ZY} \end{array}\right\} \xrightarrow{\text{⑮}} \overline{C'Y} \cong \overline{ZY} \xrightarrow{\text{⑯}} \triangle C'ZY \text{ is isos.} \\ \text{⑦}\ m \text{ bisects } \angle C'YZ \end{array}\right\} \xrightarrow{\text{⑰}} M_m(C') = Z \\ \text{⑱}\ M_m(Y) = Y \end{array}\right\} \xrightarrow{\text{⑲}}$$

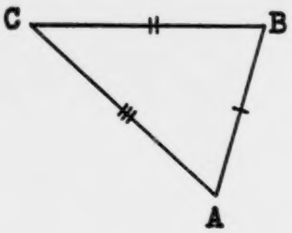
$$\xrightarrow{\text{⑲}} M_m(\triangle A'YC') = \triangle XYZ$$

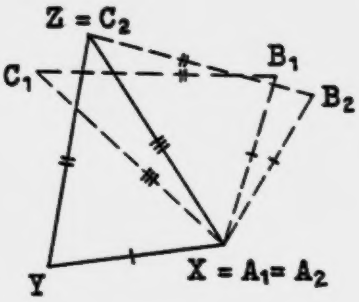
$\therefore M_m \circ T(\triangle ABC) = \triangle XYZ$ is an isometry and $\triangle ABC \cong \triangle XYZ$

Theorem 16-7: If the three sides of one triangle are congruent to the three sides of another triangle, the triangles are congruent.

Given: $\overline{AB} \cong \overline{XY}$
$\overline{BC} \cong \overline{YZ}$
$\overline{AC} \cong \overline{XZ}$

Prove: $\triangle ABC \cong \triangle XYZ$





Argument

Translate $\triangle ABC$ so that A maps onto X. $T(\triangle ABC) = \triangle XB_1C_1$
Rotate $\triangle XB_1C_1$ with center X so that C_1 maps onto Z, since $XC_1 = XZ$.

$R_X(\triangle XB_1C_1) = \triangle XB_2Z$

$XYZB_2$ is a kite since $XY = XB_2$ and $YZ = B_2Z$
$\overleftrightarrow{XZ}$ is a symmetry line for the kite by thm 16-5.

Reflect $\triangle XB_2Z$ in $\overleftrightarrow{XZ}$

$M_{\overleftrightarrow{XZ}}(X) = X$, $M_{\overleftrightarrow{XZ}}(B_2) = Y$, $M_{\overleftrightarrow{XZ}}(Z) = Z$

$M_{\overleftrightarrow{XZ}}(\triangle XB_2Z) = \triangle XYZ$

$\therefore M_{\overleftrightarrow{XZ}} \circ R_X \circ T(\triangle ABC) = \triangle XYZ$ is an isometry and $\triangle ABC \cong \triangle XYZ$

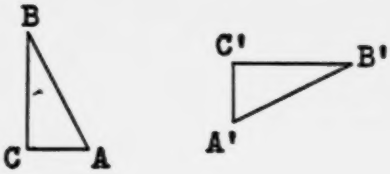
You will notice that the word correspondence is not used in either of these theorems. It is still present, of course, in a different language, as the mapping of a preimage onto its image establishes the correspondence.

EXERCISE SET 16.4

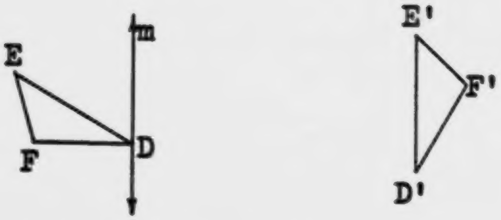
1. In all previous exercises in this chapter involving a transformation, has either the shape or size of the preimage been changed under the transformation?

2. Copy the figure and draw $\triangle A_1B_1C_1$ to illustrate that $T \circ R_A(\triangle ABC) = \triangle A'B'C'$.

3. Recopy the figure and draw $\triangle A_2B_2C_2$ to illustrate that $R_{A'} \circ T(\triangle ABC) = \triangle A'B'C'$.



4. Copy the figure and draw $\triangle D_1E_1F_1$ and $\triangle D_2E_2F_2$ to illustrate that $R_{E'} \cdot T \cdot M_m(\triangle DEF) = \triangle D'E'F'$



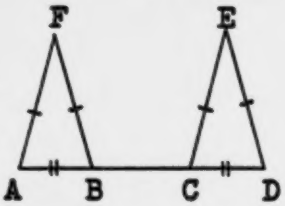
5. If two triangles are congruent and their vertices have the same orientation, how many reflections are needed in an isometry that maps one onto the other?

6. If two triangles are congruent and their vertices have reverse orientation, how many reflections are needed in an isometry that maps one onto the other?

7. Write an isometry that could be used in the proof of each of the following. You may introduce a reflecting line if necessary.

a) $\triangle ABF \cong \triangle CDE$

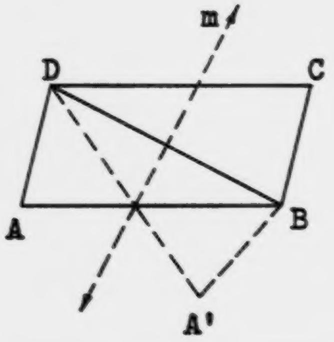
b) $\triangle ABF \cong \triangle DCE$



8. Write the reasons for statements ③, ⑧, and ⑫ in the proof of theorem 16-6.

9. Complete the following proof.

Given: $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{CB}$
Prove: $\triangle ABD \cong \triangle CDB$



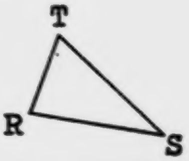
Argument

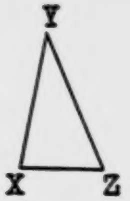
① Draw the perpendicular bisector, m, of $\overline{BD}$.
② Reflect $\triangle ABD$ in m. $M_m(\triangle ABD) = \triangle A'DB$.

(What can you prove about A'BCD? Use thm 16-7 as a guide.)

*10. Place the triangles as shown and develop an isometry to prove the following.

Given: $\angle R \cong \angle X$, $\angle T \cong \angle Z$
$\overline{RT} \cong \overline{XZ}$
Prove: $\triangle RST \cong \triangle XYZ$



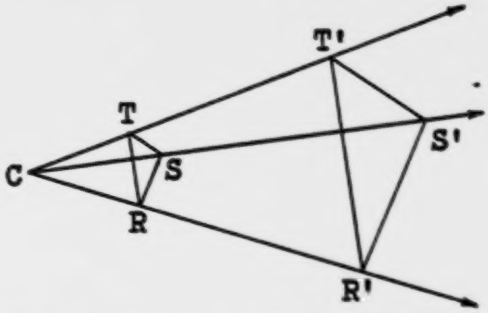


(Suggestion: Draw the perpendicular bisector of $\overline{RX}$ and reflect $\triangle RST$ in that line.)

Size Transformations

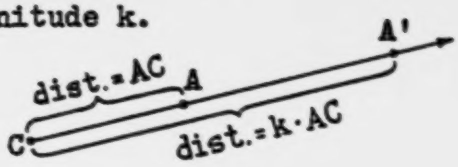
Now we'll consider a transformation in which the image is not necessarily the same size as the preimage.

Let $\triangle RST$ be a given triangle and let C be any point in the plane of $\triangle RST$. On $\overrightarrow{CR}$, $\overrightarrow{CS}$, and $\overrightarrow{CT}$, let R', S', and T' be points such that $CR' = 3 \cdot CR$, $CS' = 3 \cdot CS$, and $CT' = 3 \cdot CT$. Then $\triangle RST$ is mapped onto $\triangle R'S'T'$ by a size transformation or dilation.



Definition: Let C be a point and k be a positive real number. For any point A, let S(A) be a point A' on $\overrightarrow{CA}$ whose distance from C is k times the distance of A from C, $A'C = k \cdot AC$. Then S is the size transformation with magnitude k and center C.

We will write $S_c(A) = A'$ with magnitude k.
A' is the image of A.
A is the preimage of A'.

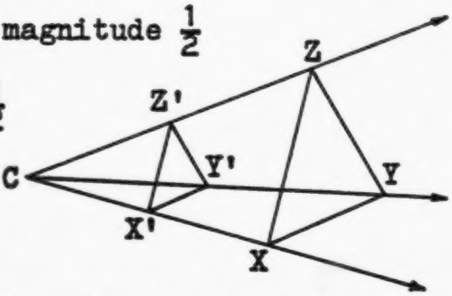


The size transformation of a figure, such as $\triangle RST$ above, is the set of all image points of the figure.

$S_c(\triangle RST) = \triangle R'S'T'$ with magnitude 3.

A size transformation with magnitude $\frac{1}{2}$ is illustrated here.

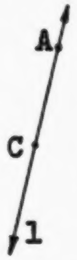
$S_C(\Delta XYZ) = \Delta X'Y'Z'$ with $k = \frac{1}{2}$



Since k is any positive real number, we can further classify size transformations as follows.

If $k > 1$, the size transformation is an expansion.
If $0 < k < 1$, the size transformation is a contraction.
If $k = 1$, the size transformation is the identity and $S(A) = A$.

If C is the center of a size transformation, the image of C is defined to be itself. The image of any line containing C is that line. The image of any ray with C as endpoint is that ray.



$S_C(C) = C$
$S_C(1) = 1$
$S_C(\overline{CA}) = \overline{CA}$

We can see that a size transformation may produce an image with a different size but with the same shape. We will accept the following postulates for size transformations.

Postulate 19: The image of a line is a line.

Postulate 20: The image of an angle is an angle of the same measure.

Postulate 21: The betweenness property of points is preserved.

Postulate 22: The orientation of the vertices of a convex polygon is preserved. (Both clockwise or both counterclockwise)

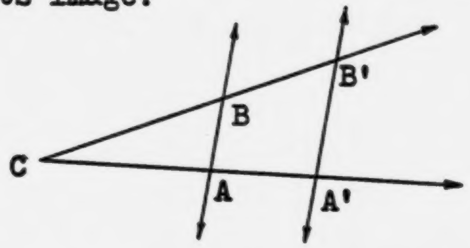
Posyulate 23: Under a size transformation of magnitude k, the distance between image points is k times the distance between their preimages.

We can now consider the following important theorems.

Theorem 16-8: Under a size transformation, a line is parallel to its image.

Given: $S_C(\overleftrightarrow{AB}) = \overleftrightarrow{A'B'}$

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ S_C(\overleftrightarrow{AB}) = \overleftrightarrow{A'B'} \\ \textcircled{2}\ S_C(\overrightarrow{CA}) = \overrightarrow{CA'} \end{array}\right\} \xrightarrow{\textcircled{3}} S_C(\angle CAB) = \angle CA'B' \xrightarrow{\textcircled{4}} m\angle CAB = m\angle CA'B' \xrightarrow{\textcircled{5}}$$

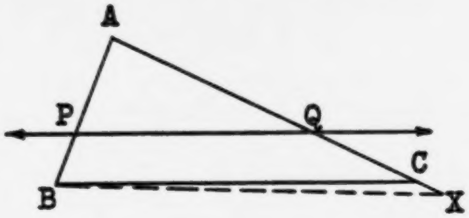
$$\xrightarrow{\textcircled{5}} \overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$$

$\therefore\ S_C(\overleftrightarrow{AB}) = \overleftrightarrow{A'B'} \longrightarrow \overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$

Theorem 16-9: If a line is parallel to side $\overline{BC}$ of $\triangle ABC$ and intersects the other two sides in distinct points, P and Q, there is a size transformation S with $S_A(\triangle APQ) = \triangle ABC$.

Given: $\overleftrightarrow{PQ} \parallel \overline{BC}$

Prove: There exists a size transformation S with $S_A(\triangle APQ) = \triangle ABC$



Argument

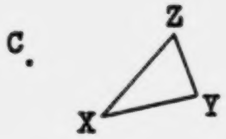
Let S_A be a size transformation of $\overleftrightarrow{PQ}$ such that the image of P is B and the image of Q is X. $S_A(\overleftrightarrow{PQ}) = \overleftrightarrow{BX}$
Then $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BX}$ by thm 16-8.
But $\overleftrightarrow{PQ} \parallel \overleftrightarrow{BC}$ is given.
Thus $\overleftrightarrow{BX}$ and $\overleftrightarrow{BC}$ are the same line by the parallel postulate, and X and C are the same point.

$S_A(A) = A,\ S_A(P) = B,\ S_A(Q) = C$

$\therefore\ S_A(\triangle APQ) = \triangle ABC$

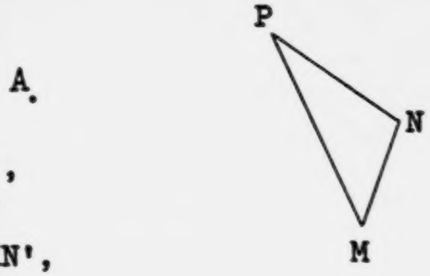
EXERCISE SET 16.5

1. Copy the figure and with C as center and $k = 4$, construct $\Delta X'Y'Z'$ such that $S_C(\Delta XYZ) = \Delta X'Y'Z'$.



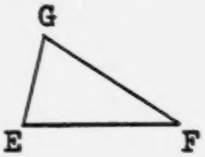
2. Measure XY, YZ, ZX, X'Y', Y'Z', and Z'X'. Compare X'Y' to XY, Y'Z' to YZ, Z'X' to ZX.
Compare $m\angle X$ to $m\angle X'$, $m\angle Y$ to $m\angle Y'$, $m\angle Z$ to $m\angle Z'$.
This size transformation is called ______.

3. Copy the figure and with A as center and $k = \frac{3}{4}$, construct $\Delta M'N'P'$ such that $S_A(\Delta MNP) = \Delta M'N'P'$.



4. Measure MN, NP, PM, M'N', N'P', and P'M'. Compare M'N' to MN, N'P' to NP, P'M' to PM.
Compare $m\angle M$ to $m\angle M'$, $m\angle N$ to $m\angle N'$, $m\angle P$ to $m\angle P'$.
This size transformation is called ______.

5. Copy the figure and with E as center and $k = 3$, construct $\Delta EF'G'$ such that $S_E(\Delta EFG) = \Delta EF'G'$



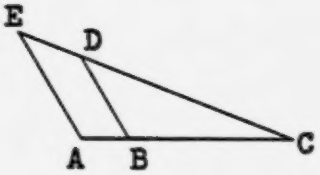
6. Compare F'G' to FG.
Compare $m\angle EFG$ to $m\angle EF'G'$ and $m\angle EGF$ to $m\angle EG'F'$.
Is it necessary that $\overline{FG} \parallel \overline{F'G'}$?

7. If $S_C(\Delta JKL) = \Delta JKL$, then $k =$ ___, and the size transformation is called ______.

8. Does a size transformation preserve or reverse the orientation of vertices of a polygon?

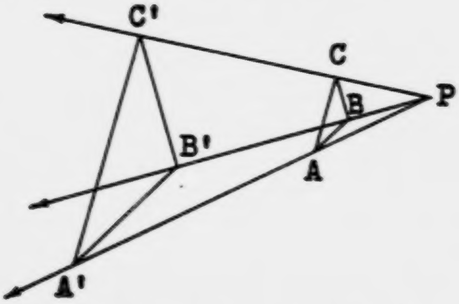
9. By theorem 16-8, $S_X(l) = l' \longrightarrow$ ______.

10. By theorem 16-9, $\overline{BD} \parallel \overline{AE} \longrightarrow$ ______.



Ratios and Proportions

Postulate 23 would be stating the following for the figure shown.



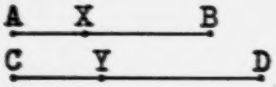
If $S_P(\Delta ABC) = \Delta A'B'C'$ with $k = 3$, then $A'B' = 3 \cdot AB$, $B'C' = 3 \cdot BC$, and $C'A' = 3 \cdot CA$.

Another way of saying that the length of A'B' is 3 times the length of AB is to say that the ratio of A'B' to AB is 3 to 1. We write this as $\frac{A'B'}{AB} = \frac{3}{1}$. The equation that says two ratios are equal is a proportion. If we state this ratio for all pairs of sides, we have $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = \frac{3}{1}$. Any such statement that says two or more ratios are equal is called a proportionality and the value of each ratio is called the proportionality constant. In this case the proportionality constant is 3. As you can see, the proportionality constant is the same as the magnitude of the transformation.

Of course, if the reciprocals of each ratio are written, the proportionality is still true with the proportionality constant $\frac{1}{k}$.

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'} = \frac{1}{3}$$

Some theorems involving proportions require the following definition.



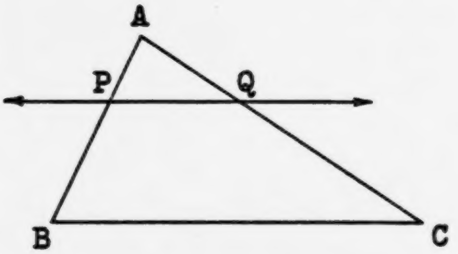
If X and Y are points on $\overline{AB}$ and $\overline{CD}$ as shown, and if $\frac{AX}{XB} = \frac{CY}{YD}$, the segments $\overline{AB}$ and $\overline{CD}$ are said to be divided proportionally. The segments of $\overline{AB}$ are said to be proportional to the segments of $\overline{CD}$.

Since proportions are equations, the proofs of some theorems involving proportions are mostly algebraic in nature.

Theorem 16-10: If a line is parallel to one side of a triangle and intersects the other two sides in distinct points, it divides them proportionally.

Given: $\overleftrightarrow{PQ} \parallel \overline{BC}$

Prove: $\frac{AP}{PB} = \frac{AQ}{QC}$



Argument

$$① \; \overleftrightarrow{PQ} \parallel \overline{BC} \xrightarrow{②} S_A(\triangle APQ) = \triangle ABC \begin{array}{l} \xrightarrow{③} AB = k\cdot AP \\ \xrightarrow{④} AC = k\cdot AQ \end{array} \Bigg\} \xrightarrow{⑤} k = \frac{AB}{AP} = \frac{AC}{AQ} \longrightarrow$$

$$\longrightarrow \ldots \xrightarrow{⑥} \frac{AP + PB}{AP} = \frac{AQ + QC}{AQ} \longrightarrow \ldots \xrightarrow{⑦} 1 + \frac{PB}{AP} = 1 + \frac{QC}{AQ} \xrightarrow{⑧}$$

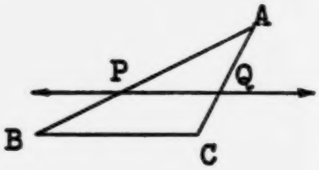
$$\xrightarrow{⑧} \frac{PB}{AP} = \frac{QC}{AQ} \xrightarrow{⑨} \frac{AP}{PB} = \frac{AQ}{QC}$$

$$\therefore \; \overleftrightarrow{PQ} \parallel \overline{BC} \longrightarrow \frac{AP}{PB} = \frac{AQ}{QC}$$

Theorem 16-11: If a line intersects two sides of a triangle and cuts off segments proportional to these sides, then the line is parallel to the third side.

Given: $\frac{AP}{AB} = \frac{AQ}{AC}$

Prove: $\overleftrightarrow{PQ} \parallel \overline{BC}$



Argument

Let S_A be a size transformation such that $k = \frac{AP}{AB} = \frac{AQ}{AC}$.

Then $S_A(B) = P$ and $S_A(C) = Q$

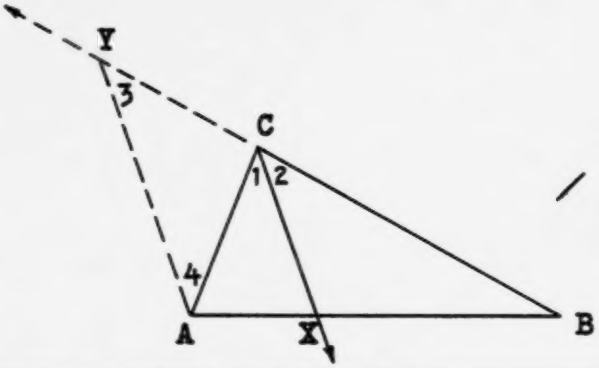
$\therefore$ $S_A(\overleftrightarrow{BC}) = \overleftrightarrow{PQ}$ and $\overleftrightarrow{PQ} \parallel \overline{BC}$ by thm 16-8.

One thing leads to another and theorem 16-10 allows us to prove an interesting theorem about proportions without reference to transformations.

Theorem 16-12: An angle bisector of a triangle intersects the opposite side in a point which divides it into segments whose lengths are proportional to the lengths of the other two sides.

Given: $\overrightarrow{CX}$ bisects $\angle ACB$

Prove: $\frac{AX}{XB} = \frac{CA}{CB}$



Argument

---

① Extend $\overrightarrow{BC}$ beyond C.

② Draw a line through A and parallel to $\overrightarrow{CX}$, intersecting $\overrightarrow{BC}$ in a point. Call it Y. ($\overline{YA} \parallel \overrightarrow{CX}$)

$$\left.\begin{array}{l} \textcircled{2}\ \overline{YA} \parallel \overline{CX} \begin{array}{l} \xrightarrow{\textcircled{3}} \angle 1 \cong \angle 4 \\ \xrightarrow{\textcircled{4}} \angle 2 \cong \angle 3 \end{array} \\ \qquad\qquad\quad \textcircled{5}\ \angle 1 \cong \angle 2 \end{array}\right\} \longrightarrow \cdots \xrightarrow{\textcircled{6}} \left.\begin{array}{r} \angle 4 \cong \angle 3 \xrightarrow{\textcircled{7}} YC = CA \\ \textcircled{2}\ \overline{YA} \parallel \overrightarrow{CX} \xrightarrow{\textcircled{8}} \frac{AX}{XB} = \frac{YC}{CB} \end{array}\right\} \xrightarrow{\textcircled{9}}$$

$$\xrightarrow{\textcircled{9}} \frac{AX}{XB} = \frac{CA}{CB}$$

$\therefore\ \overrightarrow{CX}$ bisects $\angle ACB \longrightarrow \frac{AX}{XB} = \frac{CA}{CB}$

EXERCISE SET 16.6

1. If $S_C(\Delta RST) = \Delta R'S'T'$, write each of the following as a ratio of two positive integers in simplest form for the given value of k.

a) $k = 5, \frac{R'S'}{RS} =$

b) $k = \frac{2}{3}, \frac{S'T'}{ST} =$

c) $k = 3\frac{1}{2}, \frac{R'T'}{RT} =$

d) $k = 2, \frac{SR}{S'R'} =$

e) $k = .25, \frac{T'R'}{TR} =$

f) $k = \frac{3}{7}, \frac{ST}{S'T'} =$

g) $k = .1, \frac{S'T'}{ST} =$

h) $k = .7, \frac{RS}{R'S'} =$

2. If $S_P(\Delta ABC) = \Delta A'B'C'$, write the value of k determined by each of the following proportions.

a) $\frac{A'B'}{AB} = \frac{6}{2}$

b) $\frac{A'C'}{AC} = \frac{3}{4}$

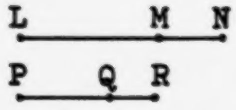
c) $\frac{BC}{B'C'} = \frac{5}{1}$

d) $\frac{BA}{B'A'} = \frac{2}{3}$

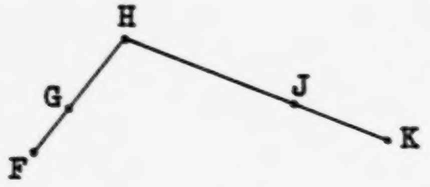
3. If $S_M(\Delta XYZ) = \Delta X'Y'Z'$ and $\frac{X'Y'}{XY} = \frac{2}{5}$, then $\frac{2}{5}$ as a ratio is called the ___________.

4. For each figure write a proportion based on the fact that two segments are divided proportionally.

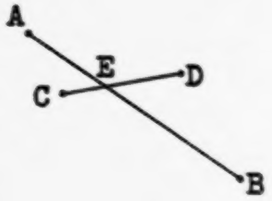
a)



b)



c)



Since a proportion is an equation, the following property is quite useful. We assume none of the variables is zero.

$$\frac{a}{b}=\frac{c}{d} \longrightarrow \frac{a}{b}\cdot bd=\frac{c}{d}\cdot bd \longrightarrow ad=bc$$

Example: $\frac{2}{3}=\frac{4}{6} \longrightarrow 2\cdot 6=3\cdot 4$

We can use this property to find the missing part or parts of a proportion.

Examples: $\frac{x}{8}=\frac{9}{12} \longrightarrow 12x=72 \longrightarrow x=6$

$$\frac{3}{4}=\frac{7}{x} \longrightarrow 3x=28 \longrightarrow x=\frac{28}{3} \text{ or } 9\tfrac{1}{3}$$

$$\frac{x+3}{x}=\frac{24}{15} \longrightarrow 15x+45=24x \longrightarrow 45=9x \longrightarrow$$

$$\longrightarrow x=5 \text{ and } x+3=8$$

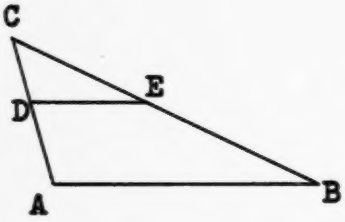
Recall the solution of a quadratic equation by factoring.

$$\frac{x+2}{8}=\frac{3}{x+4} \longrightarrow x^2+6x+8=24 \longrightarrow x^2+6x-16=0 \longrightarrow$$

$$\longrightarrow (x-2)(x+8)=0 \longrightarrow x=2 \text{ or } x=-8$$

In any example where a solution for x would result in a negative length of a segment, we must reject that solution. $\therefore x=2$.

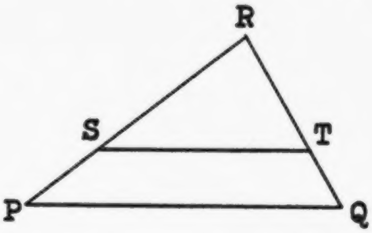
Given the figure with $\overline{DE} \parallel \overline{AB}$.



5. If CD = 3, DA = 4, CE = 5, find EB.
6. If CA = 10, CD = 4, CB = 15, find CE.
7. If BE = 8, EC = 5, DC = 3, find AD.
8. If CD = x, DA = x + 2, CE = 12, EB = 16, find CD and DA.
9. If CD = 7, DA = 9, CB = 20, find CE and EB.
10. If CE = x, EB = 15, CD = 6, DA = x + 1, find CE and DA.

For the given lengths in each exercise, state whether or not $\overline{ST} \parallel \overline{PQ}$.

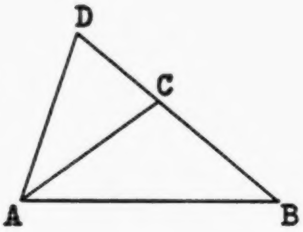
11. RS = 9, RP = 12, RT = 6, RQ = 8
12. RS = 10, SP = 4, RT = 9, TQ = 3
13. QT = 4, TR = 6, PS = $5\frac{1}{3}$, SR = 8
14. RT = 1, RQ = 2, RS = 3, SP = 3



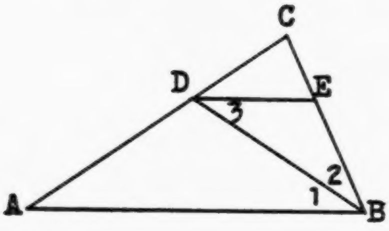
15. Write the reason for statement ⑧ in the proof of theorem 16-12.

Given the figure with $\angle DAC \cong \angle CAB$.

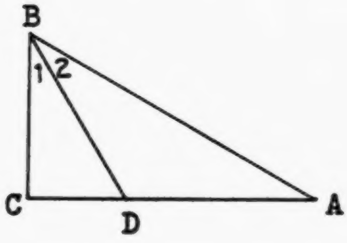
16. If BC = 9, CD = 6, AB = 12, find AD.
17. If AB = 20, AD = 15, BC = 15, find CD.
18. If AB = 10, BD = 12, DA = 8, find BC and CD.
19. If AB = x, AD = 10, BC = x - 6, CD = 7, find AB and BC.
20. If BC = 5, CD = x - 4, AB = x + 2, AD = 8, find CD and AB.



21. Given: $\overline{DE} \parallel \overline{AB}$, $\angle 1 \cong \angle 2$
Prove: $\frac{CD}{DA} = \frac{CE}{DE}$



22. Given: $\angle 1 \cong \angle A$, $\overline{BD} \cong \overline{DA}$
Prove: $\frac{CD}{BD} = \frac{BC}{BA}$



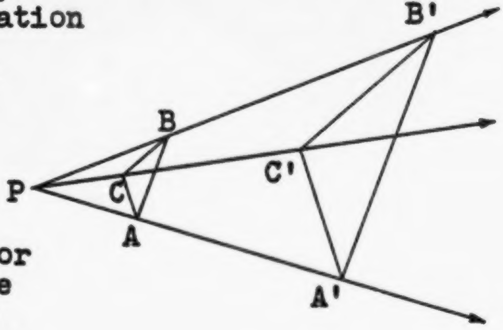
CHAPTER 17 - SIMILARITY

or I FOUND A WAY TO REDUCE!

Similarity Transformation

The word "similar" is another word that is descriptive enough to suggest its meaning in a geometrical sense. We think of things that are similar as having something in common but are not necessarily identical. We will consider geometric figures to be similar if they have the same shape but not necessarily the same size. Similarity is the characteristic of being similar. We are familiar with many examples of similarity through scale drawings, scale models, maps, and photographs. When we say that a photograph "looks like" the real thing, we express this by saying they are "proportional". That is, if a photograph shows a man standing beside a tree, the ratio of the height of the man to the height of the tree in the photo is the same as the ratio of the actual height of the man to the actual height of the tree. The ideas of ratio and proportion, as discussed in connection with size transformations, are very central ideas in the study of similarity.

We can use a size transformation to illustrate the concept of similarity. If P is the center of a size transformation with $k = 3$ which maps ΔABC onto $\Delta A'B'C'$, we know that $\Delta A'B'C'$ is an expansion of ΔABC with the same shape but not the same size. Since any isometry or size transformation will produce figures with the same shape, we give the following definitions.



A similarity transformation is a product of isometries and size transformations.

Two figures are similar if and only if there is a similarity transformation which maps one onto the other.

The symbol for "is similar to" is ~.

For the figure below,

If $S_p \cdot M_m(\text{Fig. A}) = \text{Fig. B}$, then Fig. A $\sim$ Fig. B

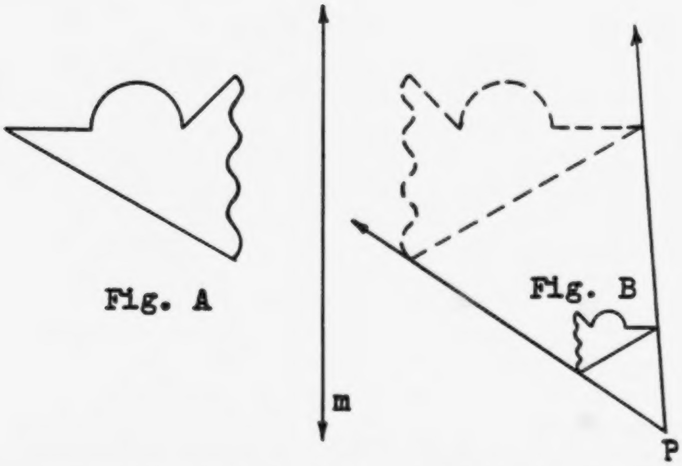
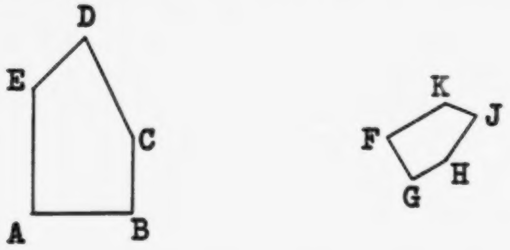
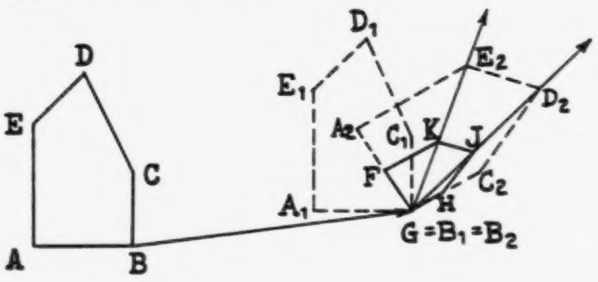


Fig. A
Fig. B
m
P

The converse part of the definition is equally important. If polygon ABCDE $\sim$ polygon FGHJK, there exists a similarity transformation which maps polygon ABCDE onto polygon FGHJK.


D
E
C
A
B
K
J
F
H
G

A possible similarity transformation is shown below in which $S_G \cdot R_G \cdot T(ABCDE) = FGHJK$


D
E
C
A
B
D_1
E_1
E_2
D_2
A_2
C_1
K
J
F
C_2
A_1
H
$G = B_1 = B_2$

If the magnitude of the size transformation in a similarity transformation is 1, of course the figures are congruent. Therefore, if two figures are congruent, they are also similar.

Theorem 17-1: Similarity of figures is reflexive.

Given: $\triangle ABC$
Prove: $\triangle ABC \sim \triangle ABC$

Any identity transformation of $\triangle ABC$ onto itself is a similarity transformation.

$\therefore \triangle ABC \sim \triangle ABC$

Theorem 17-2: Similarity of figures is symmetric.

Given: $\triangle ABC \sim \triangle DEF$
Prove: $\triangle DEF \sim \triangle ABC$

The image of $\triangle ABC$ under a similarity transformation is $\triangle DEF$. If we let $\triangle DEF$ be the preimage and reverse the order of the transformations in the similarity transformation, the image becomes $\triangle ABC$. This product is also a similarity transformation.

$\therefore \triangle ABC \sim \triangle DEF \longrightarrow \triangle DEF \sim \triangle ABC$

Theorem 17-3: Similarity of figures is transitive.

Given: $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHJ$
Prove: $\triangle ABC \sim \triangle GHJ$

$\triangle DEF$ is the image of $\triangle ABC$ under a similarity transformation and $\triangle GHJ$ is the image of $\triangle DEF$ under a similarity transformation. The product of these two similarity transformations is still a similarity transformation under which $\triangle GHJ$ is the image of $\triangle ABC$.

$$\therefore \left.\begin{array}{l}\triangle ABC \sim \triangle DEF\\ \triangle DEF \sim \triangle GHJ\end{array}\right\} \longrightarrow \triangle ABC \sim \triangle GHJ$$

EXERCISE SET 17.1

Complete the following definitions.

1. A similarity transformation is _______.
2. Two figures are similar if and only if _______.

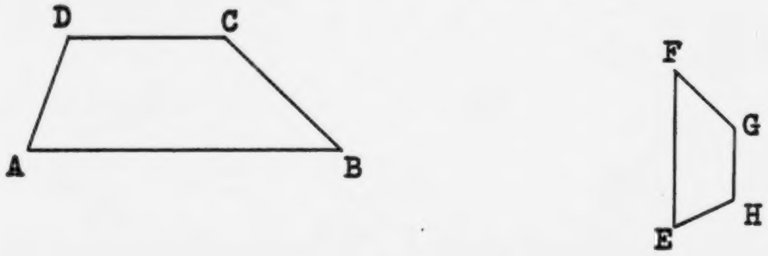
State whether or not each of the following are similar.

3. Any two rectangles
4. Any two circles
5. Any two isosceles triangles
6. Any two squares
7. Any two regular hexagons
8. Any two rhombuses
9. Any two equilateral triangles
10. Any two segments

11. A 3 inch by 5 inch photograph is enlarged so that the longer dimension of the enlargement is 12 inches. Find the shorter dimension of the enlargement.

12. The highway distance between two cities is 240 miles. If the scale of a map is 1" = 20 miles, what is the highway distance on the map between the two points representing the two cities?

13. If trap. ABCD ~ trap. EFGH, copy the figure and show a similarity transformation that will map the first onto the second.



14. Is an isometry a similarity transformation?

15. Is a size transformation a similarity transformation?

16. $\left.\begin{array}{l}\triangle LMN \sim \triangle PQR \\ \triangle PQR \sim \triangle STV\end{array}\right\} \longrightarrow ?$

Properties of Proportionalities

It was pointed out in the last chapter that if the reciprocals of each ratio in a proportionality are written, the new proportionality is still true.

(1) $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\ldots=k \longrightarrow \frac{b}{a}=\frac{d}{c}=\frac{f}{e}=\frac{h}{g}=\ldots=\frac{1}{k}$

The next property is also repeated from the previous chapter. It will be used frequently in solving proportions as there are many such problems.

(2) The cross products are equal in a proportion.

$$\frac{a}{b}=\frac{c}{d} \longrightarrow bd\cdot\frac{a}{b}=bd\cdot\frac{c}{d} \longrightarrow ad=bc$$

(3) b and c may be interchanged in the proportion $\frac{a}{b}=\frac{c}{d}$.

$$\frac{a}{b}=\frac{c}{d} \longrightarrow ad=bc \longrightarrow \frac{ad}{cd}=\frac{bc}{cd} \longrightarrow \frac{a}{c}=\frac{b}{d}$$

Other interchanges are also possible as long as the cross products always yield $ad=bc$.

(4) The ratio of the sum of the numerators to the sum of the denominators in a proportionality equals the proportionality constant.

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=k$, then $a=kb$, $c=kd$, $e=kf$, $g=kh$.

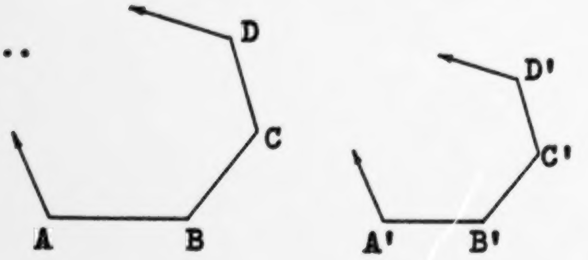
$$\therefore \frac{a+c+e+g}{b+d+f+h}=\frac{kb+kd+kf+kh}{b+d+f+h}=\frac{k(b+d+f+h)}{(b+d+f+h)}=k$$

If two polygons are similar, the ratio of the length of a side of one to the length of its image in the other is the proportionality constant for the polygons. This is also called the ratio of similitude. It will be understood that when we say "the ratio of two sides", we mean the ratio of the lengths of the sides.

Theorem 17-4: The ratio of the perimeters of two similar polygons is equal to the proportionality constant.

Given: $ABCD\dots \sim A'B'C'D'\dots$

Prove: $\frac{p}{p'} = \frac{AB}{A'B'}$



Argument

$$\left.\begin{array}{r} \left.\begin{array}{r} \text{①}\ p = AB + BC + CD + \dots \\ \text{②}\ p' = A'B' + B'C' + C'D' + \dots \end{array}\right\} \xrightarrow{\text{③}} \frac{p}{p'} = \frac{AB + BC + CD + \dots}{A'B' + B'C' + C'D' + \dots} \\ \text{④}\ \frac{AB + BC + CD + \dots}{A'B' + B'C' + C'D' + \dots} = \frac{AB}{A'B'} \end{array}\right\} \xrightarrow{\text{⑤}}$$

$$\xrightarrow{\text{⑤}} \frac{p}{p'} = \frac{AB}{A'B'}$$

$\therefore ABCD\dots \sim A'B'C'D'\dots \longrightarrow \frac{p}{p'} = \frac{AB}{A'B'}$

EXERCISE SET 17.2

Complete each of the following by using the property or theorem named.

1. Property 1: $\frac{AB}{BC} = \frac{RS}{ST} \longrightarrow$
2. Property 2: $\frac{XY}{YZ} = \frac{PY}{YQ} \longrightarrow$
3. Property 2: $\frac{PX}{PY} = \frac{PY}{PZ} \longrightarrow$
4. Property 3: $\frac{MN}{RS} = \frac{EF}{GH} \longrightarrow$
5. Property 4; $\frac{CD}{KL} = \frac{DE}{LM} \longrightarrow$
6. Property 4: If $k = \frac{2}{3}$, $\frac{AB}{WX} = \frac{BC}{XY} = \frac{CD}{YZ} \longrightarrow$
7. Theorem 17-4: $\left.\begin{array}{r} \triangle ABC \sim \triangle DEF \\ \frac{AB}{DE} = \frac{5}{2} \end{array}\right\} \longrightarrow$

State whether each of the following is true or false. Assume $x \neq 0$, $y \neq 0$, $x \neq y$.

8. $\frac{5}{8} = \frac{2\frac{1}{2}}{4}$

9. $\frac{8}{15} = \frac{12}{22}$

10. $\frac{9}{\frac{3}{4}} = \frac{48}{4}$

11. $\frac{.04}{.0004} = \frac{50}{.5}$

12. $\frac{4x}{9x} = \frac{8y}{18y}$

13. $\frac{x}{x^2} = \frac{y}{xy}$

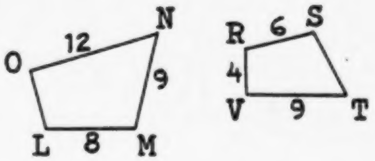
14. $\frac{3x}{5x^2} \quad \frac{6y}{10y^2}$

15. $\frac{x+2}{2x} = \frac{2x+4}{4x}$

16. Write the reason for statement ④ in the proof of theorem 17-4.

17. If LMNO ~ RSTV, what is the proportionality constant?

18. What is the ratio of the perimeter of LMNO to the perimeter of RSTV?



Similar Triangles

We will be concerned now only with similar triangles. Recalling the properties that are preserved by both isometries and size transformations, and therefore by similarity transformations, we can conclude the following for similar triangles.

1. Corresponding angles of similar triangles are congruent.
2. Corresponding sides of similar triangles are proportional.

It is these properties that will receive most of our attention in a study of similar triangles. If ΔABC can be mapped onto ΔDEF by a similarity transformation with magnitude k, such that $AB = k \cdot DE$, then we can write the following biconditional.

$$\Delta ABC \sim \Delta DEF \longleftrightarrow \begin{cases} \angle A = \angle D \\ \angle B = \angle E \\ \angle C = \angle F \\ \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = k \end{cases}$$

Once again, the mapping establishes a correspondence of preimage to image and we can identify corresponding vertices and corresponding sides of similar triangles just as we did for congruent triangles using c for "corresponds to" as before.

$$\Delta ABC \sim \Delta DEF \longrightarrow \begin{cases} A \; c \; D \\ B \; c \; E \\ C \; c \; F \\ \overline{AB} \; c \; \overline{DE} \\ \overline{BC} \; c \; \overline{EF} \\ \overline{CA} \; c \; \overline{FD} \end{cases}$$

This emphasizes the fact that a proportionality can be written directly from a statement of similarity without a figure.

$$\Delta DQX \sim \Delta MJR \longrightarrow \frac{DQ}{MJ} = \frac{QX}{JR} = \frac{DX}{MR}$$

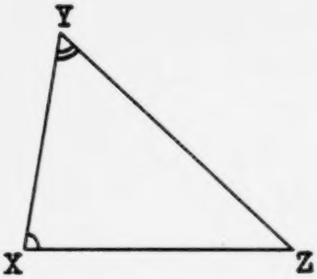
Triangle Similarity Theorems

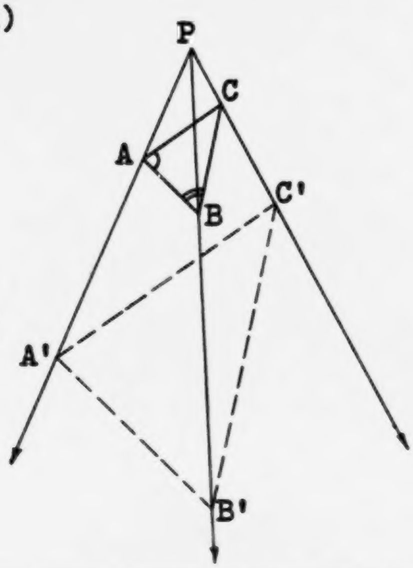
Just as with congruent triangles, we are interested in the minimum conditions that are sufficient to prove triangles similar.

Theorem 17-5: If two angles of one triangle are congruent to two angles of another triangle, the triangles are similar. (AA similarity theorem)

Given: $\angle A \cong \angle X$
$\angle B \cong \angle Y$

Prove: $\triangle ABC \sim \triangle XYZ$





Argument

① Select a point P as the center of a size transformation S with magnitude k such that $k = \frac{XY}{AB}$.

$$\left.\begin{array}{r} \text{①}\ S_P(\triangle ABC) = \triangle A'B'C' \text{ with } k = \frac{XY}{AB} \xrightarrow{\text{②}} \frac{A'B'}{AB} = \frac{XY}{AB} \xrightarrow{\text{③}} A'B' = XY \\ \left.\begin{array}{r} \text{④}\ \angle A \cong \angle X \\ \text{⑤}\ \angle A \cong \angle A' \end{array}\right\} \xrightarrow{\text{⑥}} \angle A' \cong \angle X \\ \left.\begin{array}{r} \text{⑦}\ \angle B \cong \angle Y \\ \text{⑧}\ \angle B \cong \angle B' \end{array}\right\} \xrightarrow{\text{⑨}} \angle B' \cong \angle Y \end{array}\right\} \xrightarrow{\text{⑩}}$$

$$\xrightarrow{\text{⑩}} \triangle A'B'C' \cong \triangle XYZ$$

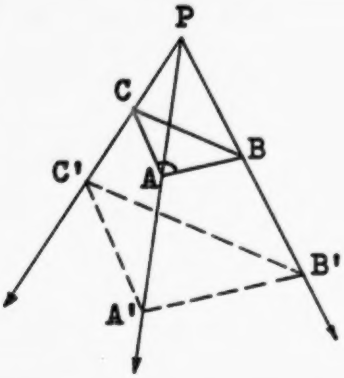
By definition of congruence, there is an isometry H such that $H(\triangle A'B'C') = \triangle XYZ$

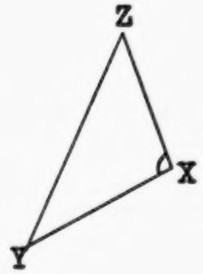
$\therefore$ $H \circ S_P(\triangle ABC) = \triangle XYZ$ is a similarity transformation and $\triangle ABC \sim \triangle XYZ$.

Theorem 17-6: If two sides of one triangle are proportional to two sides of another triangle and the included angles are congruent, the triangles are similar. (SAS similarity theorem)

Given: $\frac{AB}{XY}=\frac{AC}{XZ}$
$\angle A \cong \angle X$

Prove: $\triangle ABC \sim \triangle XYZ$





Argument

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① $\frac{AB}{XY}=\frac{AC}{XZ}\xrightarrow{②}\frac{XY}{AB}=\frac{XZ}{AC}$

③ Select a point P as center of a size transformation with magnitude k such that $k=\frac{XY}{AB}=\frac{XZ}{AC}$.

$$\left.\begin{array}{r} \text{③ } S_P(\triangle ABC)=\triangle A'B'C' \text{ with } k=\frac{XY}{AB}\xrightarrow{④}\frac{A'B'}{AB}=\frac{XY}{AB}\xrightarrow{⑤}A'B'=XY \\ \text{③ } k=\frac{XZ}{AC}\xrightarrow{⑥}\frac{A'C'}{AC}=\frac{XZ}{AC}\xrightarrow{⑦}A'C'=XZ \\ \left.\begin{array}{r}\text{⑧ } \angle A=\angle A' \\ \text{⑨ } \angle A=\angle X\end{array}\right\}\xrightarrow{⑩}\angle A'=\angle X\end{array}\right\}\xrightarrow{⑪}$$

$\xrightarrow{⑪}\triangle A'B'C'=\triangle XYZ$

By definition of congruence, there is an isometry H such that $H(\triangle A'B'C')=\triangle XYZ$

$\therefore$ $H\circ S_P(\triangle ABC)=\triangle XYZ$ is a similarity transformation and $\triangle ABC\sim\triangle XYZ$.

Theorem 17-7: If the three sides of one triangle are proportional to the three sides of another triangle, the triangles are similar. (SSS similarity theorem)

Given: $\frac{AB}{XY}=\frac{AC}{XZ}=\frac{BC}{YZ}$

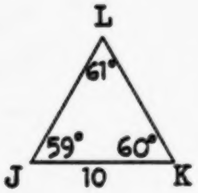
Prove: $\triangle ABC\sim\triangle XYZ$

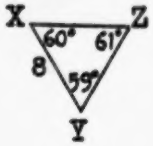
The proof is similar to the preceding proofs and is left as an exercise.

EXERCISE SET 17.3

Complete

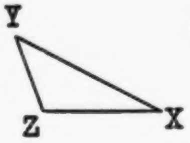
1. Corresponding angles of similar triangles are ______.
2. Corresponding sides of similar triangles are ______.
3. $\triangle ARX \sim \triangle BYT \longrightarrow \frac{RA}{?} = \frac{AX}{?} = \frac{RX}{?}$
4. Write the statement that says the triangles shown are similar. Can you find the ratio of similitude from the information on the figure?

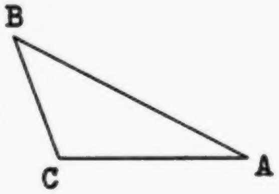




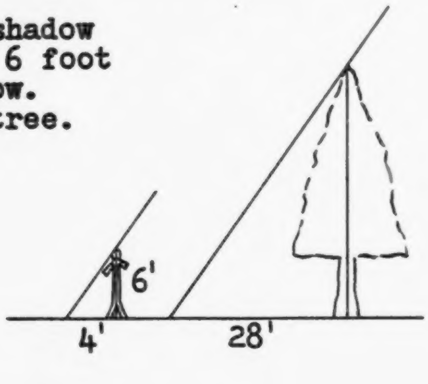
5. Write the reasons for statements ②, ⑤, ⑧, and ⑩ in the proof of theorem 17-5.
6. Write the reasons for statements ② and ⑪ in the proof of theorem 17-6.
7. Prove theorem 17-7. (Draw the triangles randomly placed and use the two previous theorems as a guide.)

Given $\triangle ABC \sim \triangle XYZ$.

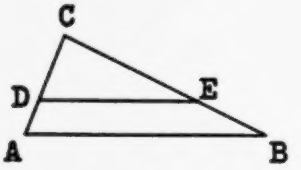




8. If $AC = 12$, $XZ = 10$, $BC = 6$, find YZ.
9. If $AB = 15$, $CA = 9$, $YZ = 4$, $ZX = 6$, find BC and XY.
10. If $BC = x$, $BA = x + 5$, $YZ = 9$, $YX = 12$, find BC and BA.
11. If $m\angle A = 35$, $m\angle Z = 105$, find $m\angle X$, $m\angle Y$, and $m\angle C$.
12. If $AB = 45$, $YZ = 20$, $BC = XY$, find BC and XY.
13. A tree casts a 28 foot shadow at the same time that a 6 foot man casts a 4 foot shadow. Find the height of the tree.



14. Given: $\overline{DE} \parallel \overline{AB}$
Prove: $\triangle ABC \sim \triangle DEC$



Use the figure of exercise 14 for each of the following.

15. If $CD = 9$, $CA = 12$, $DE = 15$, find AB.

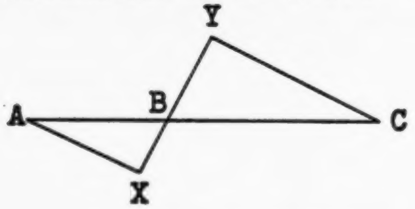
16. If $CE = 6$, $EB = 2$, $AB = 10$, find DE.

17. If $AD = 5$, $DC = 7$, $CE = 11$, find EB.

18. If $CD = x$, $DA = 2$, $DE = 2x$, $AB = 18$, find CD.

We proved triangles congruent in order to prove segments congruent. We can now prove triangles similar in order to prove segments proportional or to prove consequences of proportions.

Example: Given: $\overline{AX} \parallel \overline{YC}$
Prove: $AB \cdot YB = XB \cdot CB$

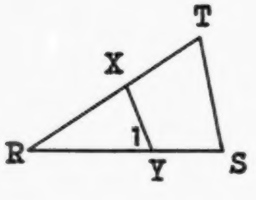


Argument

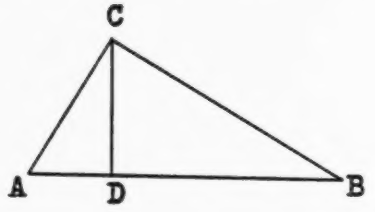
$$\text{①}\ \overline{AX} \parallel \overline{YC} \left.\begin{array}{l} \xrightarrow{\text{②}} \angle X \cong \angle Y \\ \xrightarrow{\text{③}} \angle A \cong \angle C \end{array}\right\} \xrightarrow{\text{④}} \triangle ABX \sim \triangle CBY \xrightarrow{\text{⑤}} \frac{AB}{CB} = \frac{XB}{YB} \xrightarrow{\text{⑥}}$$

$$\xrightarrow{\text{⑥}} AB \cdot YB = XB \cdot CB$$

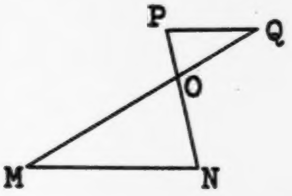
19. Given: $\angle 1 \cong \angle T$
Prove: $\frac{RS}{RT} = \frac{RX}{RY}$



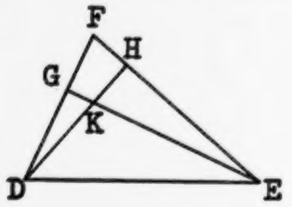
20. Given: $\overline{CD} \perp \overline{AB}$, $\frac{AD}{CD} = \frac{CD}{BD}$
Prove: $\triangle ACD \sim \triangle CBD$



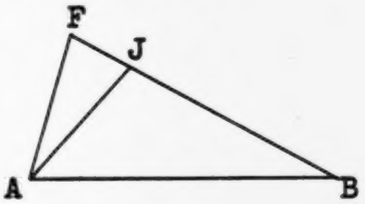
21. Given: $\angle M \cong \angle Q$
Prove: $\frac{NO}{PO} = \frac{NM}{PQ}$



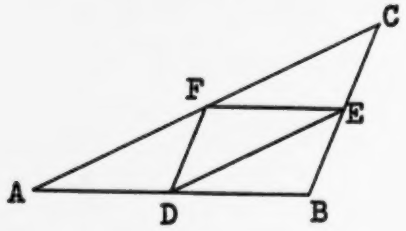
22. Given: $\overline{DH} \perp \overline{EF}$, $\overline{EG} \perp \overline{DF}$
Prove: $DG \cdot EK = EH \cdot DK$



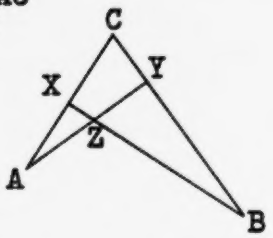
23. Given: $\overline{AB} \cong \overline{FB}$, $\overline{AF} \cong \overline{AJ}$
Prove: $\frac{FJ}{FA} = \frac{FA}{FB}$



24. Given: D, E, and F are midpoints
Prove: $\triangle DEF \sim \triangle CAB$

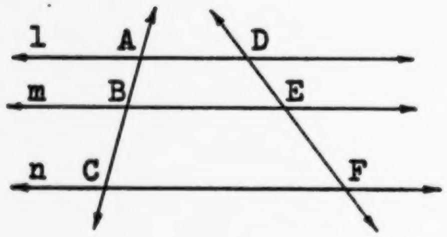


25. Given: $\overline{AY} \perp \overline{BC}$, $\overline{BX} \perp \overline{AC}$
Prove: $AC \cdot CX = BC \cdot CY$

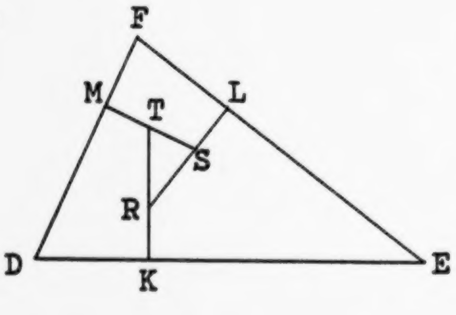


26. Given: $l \parallel m \parallel n$
Prove: $\frac{AB}{BC} = \frac{DE}{EF}$

(Hint: Draw $\overline{AF}$ or $\overline{CD}$)



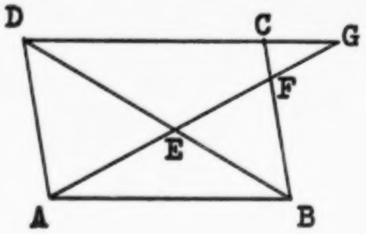
27. Given: $\overline{TK} \perp \overline{DE}$, $\overline{RL} \perp \overline{EF}$, $\overline{SM} \perp \overline{DF}$
Prove: $\triangle DEF \sim \triangle TRS$



28. Prove: If two triangles are similar, the ratio of corresponding medians equals the ratio of similitude.

29. Given: ABCD is a ▱

Prove: $(AE)^2 = EF \cdot EG$



Right Triangle Theorems

For the next set of theorems, we must give the following definition.

If $\frac{a}{b} = \frac{b}{c}$, then b is the geometric mean between a and c.

Example: Since $\frac{2}{4} = \frac{4}{8}$, then 4 is the geometric mean between 2 and 8.

In geometry, all the numbers in a proportion are positive. Under this restriction, the geometric mean of two numbers is the square root of their product.

$$\frac{a}{b} = \frac{b}{c} \longrightarrow b^2 = ac \longrightarrow b = \sqrt{ac}$$

Since some of the problems will involve square root radicals, we will briefly review the three cases in which such radicals are simplified. The laws of radicals that justify these steps are as follows.

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab} \longleftrightarrow \sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \longleftrightarrow \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Three cases involving simplifying radicals are as follows.

1. A perfect square factor of the radicand.

$$\sqrt{18}=\sqrt{9\cdot 2}=\sqrt{9}\cdot\sqrt{2}=3\sqrt{2}$$

2. The radicand is a fraction.

$$\sqrt{\frac{2}{3}}=\sqrt{\frac{2}{3}\cdot\frac{3}{3}}=\sqrt{\frac{6}{9}}=\frac{\sqrt{6}}{\sqrt{9}}=\frac{\sqrt{6}}{3}$$

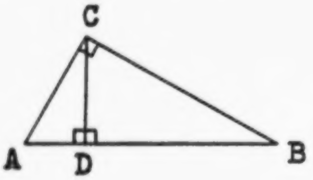
3. The denominator of a fraction is a radical.

$$\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

Theorem 17-8: The altitude to the hypotenuse of a right triangle forms two triangles that are similar to each other and to the original triangle.

Given: $\angle ACB$ is a rt. $\angle$ in $\triangle ABC$
$\overline{CD}$ is the altitude to $\overline{AB}$

Prove: $\triangle ACD \sim \triangle ABC \sim \triangle CBD$



Argument

$$\left.\begin{array}{l}
\left.\begin{array}{r} \text{①}\ \overline{CD}\perp\overline{AB} \xrightarrow{\text{②}} \angle ADC \text{ is a rt. } \angle \\ \text{③}\ \angle ACB \text{ is a rt. } \angle \end{array}\right\} \xrightarrow{\text{④}} \left.\begin{array}{r} \angle ADC \cong \angle ACB \\ \text{⑤}\ \angle DAC \cong \angle CAB \end{array}\right\} \xrightarrow{\text{⑥}} \triangle ACD \sim \triangle ABC \\
\left.\begin{array}{r} \text{①}\ \overline{CD}\perp\overline{AB} \xrightarrow{\text{⑦}} \angle BDC \text{ is a rt. } \angle \\ \text{⑧}\ \angle BCA \text{ is a rt. } \angle \end{array}\right\} \xrightarrow{\text{⑨}} \left.\begin{array}{r} \angle BDC \cong \angle BCA \\ \text{⑩}\ \angle DBC \cong \angle CBA \end{array}\right\} \xrightarrow{\text{⑪}} \triangle CBD \sim \triangle ABC
\end{array}\right\} \xrightarrow{\text{⑫}}$$

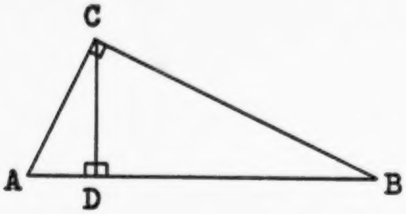
$$\xrightarrow{\text{⑫}} \triangle ACD \sim \triangle CBD$$

$$\therefore \left.\begin{array}{l} \angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC \\ \overline{CD} \text{ is the altitude to } \overline{AB} \end{array}\right\} \longrightarrow \triangle ACD \sim \triangle ABC \sim \triangle CBD$$

Three similar triangles in one figure! Think of all the proportions we can write from that. The following corollaries state some of these in words. Any reference to a segment is understood to mean the length of the segment where appropriate. The trick in the proofs is to make sure that we select the pair of similar triangles that are related to the segments in the statement we are trying to prove.

Corollary 17-8a: The altitude to the hypotenuse of a right triangle separates the hypotenuse into segments such that each leg of the right triangle is the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to that leg.

This is an awkward statement of a theorem that is more easily understood by examining the symbolic statements to be proved.



Given: $\angle ACB$ is a rt. $\angle$ in $\triangle ABC$
$\overline{CD}$ is the altitude to $\overline{AB}$

Prove: $\frac{AD}{AC} = \frac{AC}{AB}$ and $\frac{BD}{BC} = \frac{BC}{BA}$

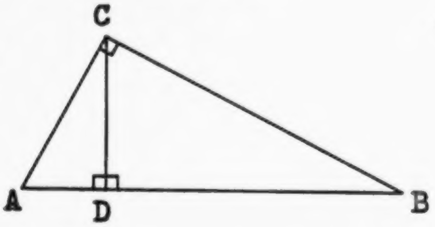
Argument

$$\left.\begin{array}{l} \textcircled{1}\ \angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC \\ \textcircled{2}\ \overline{CD} \text{ is the altitude to } \overline{AB} \end{array}\right\} \xrightarrow{\textcircled{3}} \triangle ACD \sim \triangle ABC \xrightarrow{\textcircled{4}} \frac{AD}{AC} = \frac{AC}{AB}$$

$$\left.\begin{array}{l} \textcircled{1}\ \angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC \\ \textcircled{2}\ \overline{CD} \text{ is the altitude to } \overline{AB} \end{array}\right\} \xrightarrow{\textcircled{5}} \triangle CBD \sim \triangle ABC \xrightarrow{\textcircled{6}} \frac{BD}{BC} = \frac{BC}{BA}$$

$$\therefore \left.\begin{array}{l} \angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC \\ \overline{CD} \text{ is the altitude to } \overline{AB} \end{array}\right\} \begin{array}{l} \nearrow \frac{AD}{AC} = \frac{AC}{AB} \\ \searrow \frac{BD}{BC} = \frac{BC}{BA} \end{array}$$

Corollary 17-8b: The altitude to the hypotenuse of a right triangle is the geometric mean between the segments of the hypotenuse.



Given: $\angle ACB$ is a rt. $\angle$ in $\triangle ABC$
$\overline{CD}$ is the altitude to $\overline{AB}$

Prove: $\frac{AD}{CD} = \frac{CD}{BD}$

Argument

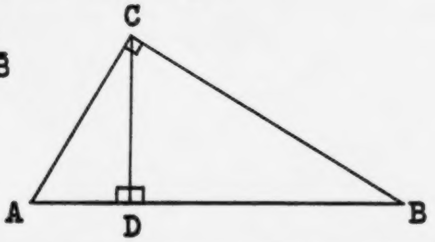
$$\left.\begin{array}{l} \textcircled{1}\ \angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC \\ \textcircled{2}\ \overline{CD} \text{ is the altitude to } \overline{AB} \end{array}\right\} \xrightarrow{\textcircled{3}} \triangle ACD \sim \triangle CBD \xrightarrow{\textcircled{4}} \frac{AD}{CD} = \frac{CD}{BD}$$

$$\therefore \left.\begin{array}{l} \angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC \\ \overline{CD} \text{ is the altitude to } \overline{AB} \end{array}\right\} \longrightarrow \frac{AD}{CD} = \frac{CD}{BD}$$

Corollary 17-8c: The product of the hypotenuse of a right triangle and the altitude to it is equal to the product of the legs.

Given: $\angle ACB$ is a rt. $\angle$ in $\triangle ABC$
$\overline{CD}$ is the altitude to $\overline{AB}$

Prove: $AB \cdot CD = AC \cdot BC$



Argument

$$\left.\begin{array}{l}\text{①}\ \angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC\\ \text{②}\ \overline{CD} \text{ is the altitude to } \overline{AB}\end{array}\right\} \xrightarrow{\text{③}} \triangle ABC \sim \triangle ACD \xrightarrow{\text{④}} \frac{AB}{AC} = \frac{BC}{CD} \xrightarrow{\text{⑤}}$$

$$\xrightarrow{\text{⑤}} AB \cdot CD = AC \cdot BC$$

$$\therefore \left.\begin{array}{l}\angle ACB \text{ is a rt. } \angle \text{ in } \triangle ABC\\ \overline{CD} \text{ is the altitude to } \overline{AB}\end{array}\right\} \longrightarrow AB \cdot CD = AC \cdot BC$$

EXERCISE SET 17.4

Find the geometric mean between each of the following pairs of numbers.

1. 3 and 27
2. 4 and 16
3. 2 and 18
4. 8 and 50
5. 3 and 5
6. x and y

Write each of the following in simplest radical form.

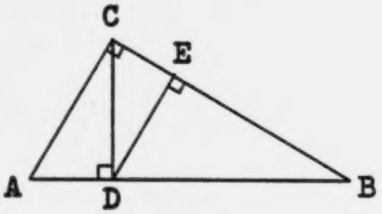
7. $\sqrt{8}$
8. $3\sqrt{50}$
9. $\frac{\sqrt{72}}{2}$
10. $\sqrt{500}$
11. $\sqrt{\frac{1}{5}}$
12. $\sqrt{\frac{7}{10}}$
13. $\sqrt{\frac{3}{4}}$
14. $\sqrt{\frac{5}{8}}$
15. $\frac{1}{\sqrt{2}}$
16. $\frac{5}{\sqrt{5}}$
17. $\frac{6}{\sqrt{12}}$
18. $\frac{\sqrt{2}}{\sqrt{3}}$

Solve each proportion for positive values of x.

19. $\frac{2}{x} = \frac{x}{32}$
20. $\frac{x}{4} = \frac{5}{x}$
21. $\frac{x}{\sqrt{2}} = \frac{\sqrt{8}}{x}$
22. $\frac{2\sqrt{5}}{x} = \frac{x}{\sqrt{5}}$

23. Write the reasons for statements ⑥, ⑪, and ⑫ in the proof of theorem 17-8.

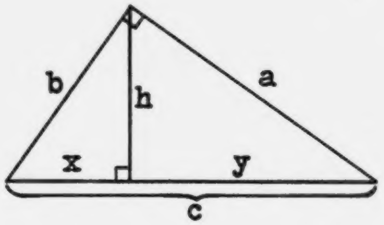
24. How many triangles are in the figure shown? Are any two of those triangles not similar?



25. Write the two conclusions of corollary 17-8a using the segment lengths shown.

26. Write the conclusion of corollary 17-8b using these segment lengths.

27. Write the conclusion of corollary 17-8c using these segment lengths.



In each of the following be sure to make use of the corollary that involves the segments of that particular exercise. Use the figure above.

28. If $c = 12$, $a = 9$, find y.

29. If $x = 3$, $y = 9$, find h.

30. If $a = 8$, $b = 6$, $c = 10$, find h.

31. If $c = 3\sqrt{6}$, $x = \sqrt{6}$, find b.

32. If $x = 4$, $h = 10$, find y.

33. If $c = 10$, $y = 8$, find a.

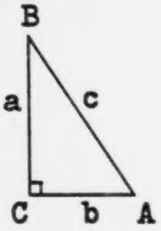
34. If $a = 2\sqrt{2}$, $b = 2\sqrt{3}$, $c = 2\sqrt{5}$, find h.

35. If $c = 2x$, $b = 5$, find x and c.

The Pythagorean Theorem

One of the highlights of any study of geometry is the Pythagorean Theorem. The fascination for this right triangle relationship has provided motivation for over 350 different proofs. Some of these involve area relationships which we will include in a later chapter. However, a proof involving the use of proportions can be given here and ties in nicely with our present study.

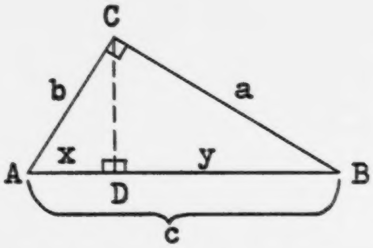
In a right triangle, it is conventional to label the vertices as shown with the vertex at the right angle as C. The lengths of the sides are then labeled with lower case letters corresponding to the capital letter at the opposite vertex. In this way, the legs will be a and b, and the hypotenuse will be c.



Theorem 17-9: (Pythagorean Theorem) In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.

Given: $\angle C$ is a rt. $\angle$ in $\triangle ABC$

Prove: $c^2 = a^2 + b^2$



Argument

① Draw $\overline{CD} \perp \overline{AB}$. Let $AD = x$ and $BD = y$.

$$\begin{tikzcd} \left.\begin{array}{r} \text{②}\ \angle C \text{ is a rt. } \angle \text{ in } \triangle ABC \\ \text{①}\ \overline{CD} \perp \overline{AB} \end{array}\right\} \arrow[r, "\text{③}"] \arrow[dr, "\text{④}"'] & \frac{y}{a} = \frac{a}{c} \arrow[r, "\text{⑤}"] & yc = a^2 \arrow[dr, "\text{⑦}"] & \\ & \frac{x}{b} = \frac{b}{c} \arrow[r, "\text{⑥}"] & xc = b^2 \arrow[r, "\text{⑦}"'] & {} \end{tikzcd}$$

$$\xrightarrow{⑦} yc + xc = a^2 + b^2 \xrightarrow{⑧} \left.\begin{array}{r} c \cdot (y + x) = a^2 + b^2 \\ ⑨\ y + x = c \end{array}\right\} \xrightarrow{⑩} c^2 = a^2 + b^2$$

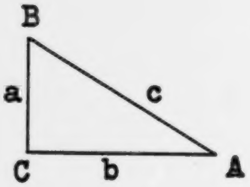
$\therefore \angle C$ is a rt. $\angle$ in $\triangle ABC \longrightarrow c^2 = a^2 + b^2$

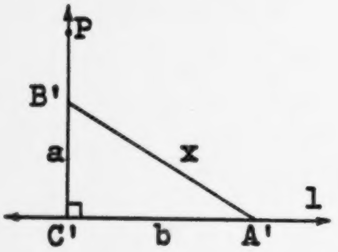
The converse of the Pythagorean Theorem gives us the conditions under which any triangle must be a right triangle.

Theorem 17-10: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, the triangle is a right triangle with the right angle opposite the longest side.

Given: $c^2 = a^2 + b^2$

Prove: $\triangle ABC$ is a rt. $\triangle$ with $\angle C$ a rt. $\angle$





Argument

① At a point C' on a line l, let $\overrightarrow{C'P} \perp l$.
② Let B' be a point on $\overrightarrow{C'P}$ such that $C'B' = CB = a$.
③ Let A' be a point on l such that $C'A' = CA = b$.
④ Let $A'B' = x$

$$①\ \overrightarrow{C'P} \perp l \xrightarrow{⑤} \angle A'C'B' \text{ is a rt. } \angle \left.\begin{array}{l} \xrightarrow{⑥} x^2 = a^2 + b^2 \\ ⑦\ c^2 = a^2 + b^2 \end{array}\right\} \xrightarrow{⑧}$$

$$\xrightarrow{⑧} x^2 = c^2 \xrightarrow{⑨} x = c \left.\begin{array}{l} \xrightarrow{⑩} \overline{A'B'} \cong \overline{AB} \\ ②\ \overline{B'C'} \cong \overline{BC} \\ ③\ \overline{A'C'} \cong \overline{AC} \end{array}\right\} \xrightarrow{⑪} \triangle ABC \cong \triangle A'B'C' \xrightarrow{⑫}$$

$$\left.\begin{array}{l} \xrightarrow{⑫} \angle ACB \cong \angle A'C'B' \\ ⑤\ \angle A'B'C' \text{ is a rt. } \angle \end{array}\right\} \xrightarrow{⑬} \angle ACB \text{ is a rt. } \angle$$

$\therefore c^2 = a^2 + b^2 \longrightarrow \triangle ABC$ is a rt. $\triangle$ with $\angle C$ a rt. $\angle$

EXERCISE SET 17.5

1. Write the reasons for all the statements in the proof of theorem 17-9.
2. Write the reasons for all the statements except ④ in the proof of theorem 17-10.
3. In right triangle ΔABC, a and b are the legs and c is the hypotenuse. Express any irrational lengths in simplest radical form.

   a) If $a = 4$, $b = 3$, find c.

   b) If $a = 8$, $c = 10$, find b.

   c) If $b = 5$, $c = 13$, find a.

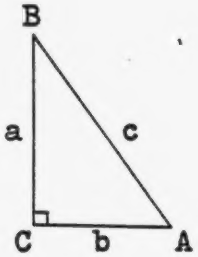
   d) If $a = 6$, $b = 4$, find c.

   e) If $b = 10$, $c = 10\sqrt{5}$, find a.

   f) If $a = 8$, $b = 4\sqrt{5}$, find c.

   g) If $b = 2\sqrt{3}$, $c = 6$, find a.

   h) If $a = 2b$, $c = 10$, find a and b.



4. State whether or not each set of numbers can be the lengths of the sides of a right triangle.

   a) $4, 5, 6$

   b) $1\frac{1}{2}, 2, 2\frac{1}{2}$

   c) $9, 12, 15$

   d) $1, 3, \sqrt{10}$

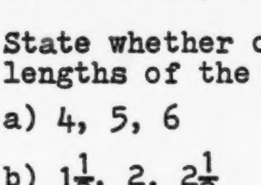
   e) $\sqrt{3}, \sqrt{4}, \sqrt{5}$

   f) $8, 15, 17$

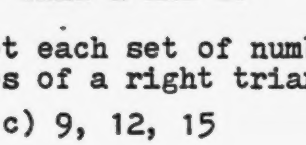
5. The length of a rectangle is 7 inches and the width is 5 inches. Find the length of a diagonal.
6. The diagonals of a rhombus have lengths 16 inches and 12 inches. Find the perimeter of the rhombus.
7. A hiker walks 5 miles north, 2 miles east, and 1 mile north. How far is he from his starting point?
8. The legs of an isosceles triangle are each 15 inches long. If the base is 10 inches long, find the length of the altitude to the base.

Find x.

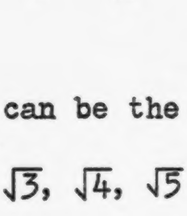
9.



10.

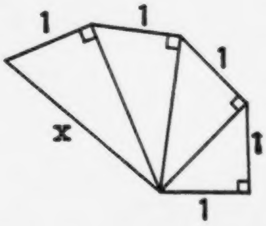


11.

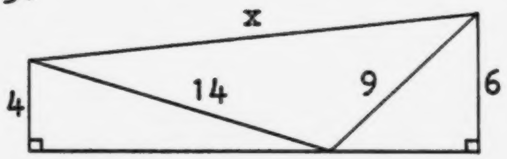


Find x.

12.



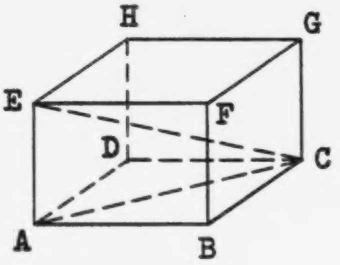
13.



14. The space figure shown is a rectangular solid in which all the faces are rectangles. $\overline{AC}$ is a diagonal of one face and $\overline{EC}$ is a diagonal of the solid. By the Pythagorean Theorem,

$$\left.\begin{array}{l}(AC)^2 = (AB)^2 + (BC)^2 \\ (EC)^2 = (AC)^2 + (AE)^2\end{array}\right\} \longrightarrow (EC)^2 = (AB)^2 + (BC)^2 + (AE)^2 \longrightarrow$$

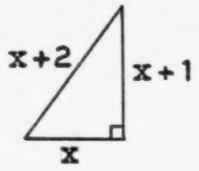
$$\longrightarrow EC = \sqrt{(AB)^2 + (BC)^2 + (AE)^2}$$



a) If $AB = 12$, $BC = 4$, $AE = 3$, find EC.

b) If $EF = 8$, $EH = 6$, $EA = 5$, find HB.

15. Any combination of integers that can be lengths of the sides of a right triangle is called a Pythagorean triple. Show that 3, 4, 5 is the only Pythagorean triple consisting of consecutive integers by applying the Pythagorean Theorem to the figure shown.



16. It can be shown that if x and y are any positive integers with $x > y$, that $x^2 - y^2$, $2xy$, $x^2 + y^2$ will be a Pythagorean triple a, b, c. For each of the values of x and y below, find a, b, and c and verify that $a^2 + b^2 = c^2$.

a) $x = 3$, $y = 2$

b) $x = 4$, $y = 1$

c) $x = 4$, $y = 3$

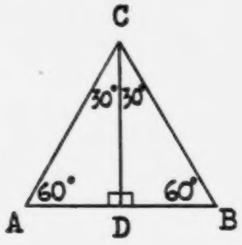
d) $x = 5$, $y = 2$

Special Right Triangles

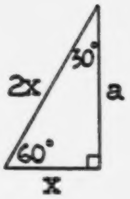
Two particular right triangles are worth special attention. They are the 30°- 60°- 90° triangle and the 45°- 45°- 90° triangle. The latter is the isosceles right triangle. Any student of drafting is familiar with these triangles as the instruments used in that activity.

Let $\triangle ABC$ be equilateral. If a perpendicular is dropped from any vertex to the opposite side, the two triangles formed are each 30°- 60°- 90° triangles.

Since $AB = 2 \cdot AD$, then $AC = 2 \cdot AD$. The shorter leg is always opposite the 30° angle and the longer leg opposite the 60° angle.

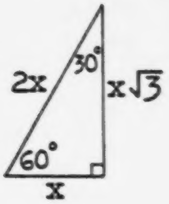


If we call the length of the shorter leg x, then the length of the hypotenuse is 2x. Let the length of the longer leg be a. Then by the Pythagorean Theorem we can write the following.

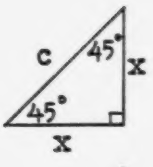


$$x^2 + a^2 = (2x)^2 \longrightarrow x^2 + a^2 = 4x^2 \longrightarrow a^2 = 3x^2 \longrightarrow a = x\sqrt{3}$$

Therefore, the three sides can all be expressed in terms of a single variable, so that if we know any one, we can find both of the other two.

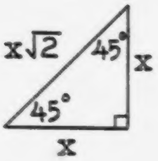


The relationships of the sides can be found even more directly for the isosceles right triangle since the legs have equal length.



By the Pythagorean Theorem,

$$c^2 = x^2 + x^2 \longrightarrow c^2 = 2x^2 \longrightarrow c = x\sqrt{2}$$



EXERCISE SET 17.6

1. Find the lengths of the hypotenuse and longer leg of a 30°-60°-90° triangle if the length of the shorter leg is

   a) 6 b) 10 c) $4\sqrt{3}$

2. Find the lengths of the shorter leg and the longer leg of a 30°-60°-90° triangle if the length of the hypotenuse is

   a) 8 b) $6\sqrt{3}$ c) $2\sqrt{6}$

3. Find the lengths of the shorter leg and hypotenuse of a 30°-60°-90° triangle if the length of the longer leg is

   a) 12 b) 3 c) $5\sqrt{3}$

4. Can the lengths of the three sides of a 30°-60°-90° triangle all be rational numbers? If so, give an example.

5. Can the lengths of the three sides of a 30°-60°-90° triangle all be irrational numbers? If so, give an example.

6. Find the length of the hypotenuse of an isosceles right triangle if the length of each leg is

   a) 5 b) $3\sqrt{2}$ c) $\sqrt{3}$

7. Find the length of each leg of a 45°-45°-90° triangle if the length of the hypotenuse is

   a) 8 b) 1 c) $4\sqrt{2}$

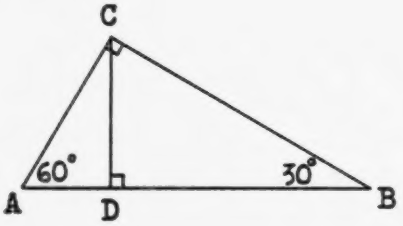
8. Can the lengths of the three sides of an isosceles right triangle all be rational numbers? If so, give an example.

9. Can the lengths of the three sides of an isosceles right triangle all be irrational numbers? If so, give an example.

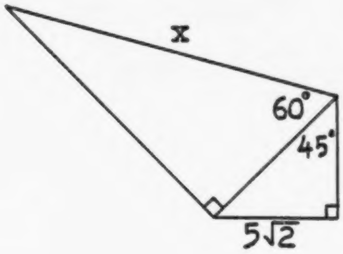
10. The length of a side of an equilateral triangle is 18. Find the length of an altitude.

11. The length of a side of a square is 14. Find the length of a diagonal.

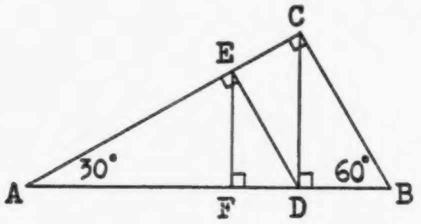
In the given figure, if the length of any one of the six segments is known, the lengths of the five remaining segments can all be found.



12. If AB = 12, find CD.
13. If BC = 8, find AD.
14. If AD = 6, find BD.
15. If AC = $10\sqrt{3}$, find DB.
16. If CD = $4\sqrt{3}$, find all remaining lengths.
17. Prove: In a 30°-60°-90° triangle the altitude to the hypotenuse divides the hypotenuse into segments whose lengths are in the ratio 1 to 3.
18. The measure of one angle of a rhombus is 120. The length of the longer diagonal is 18. Find the perimeter of the rhombus.
19. Find x.



20. Find the ratio of similitude for △AFE ~ △CDB.
21. Find the ratio of similitude for △DEF ~ △BAC.



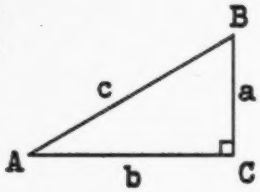
Right Triangle Trigonometry

Throughout much of the history of mathematics, trigonometry has been thought of as a useful tool in the study of astronomy, navigation, and surveying. It is all of that but it is much more, as it is thoroughly integrated with other areas of mathematics. We will present here a short unit as it relates to the right triangle, and if you study trigonometry later, you will see how this ties in with a broader interpretation of the subject.

The word trigonometry actually means " triangle measurement" which is descriptive of our study of it here, but it is not to be taken as a definition.

If we were to write every possible ratio of a pair of sides for the right triangle shown, we would write the following six ratios.

$$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{c}{a}, \frac{c}{b}, \frac{b}{a}$$



With respect to a particular acute angle of the triangle, each of these ratios has its own name and they are called the trigonometric ratios. Notice that the last three are reciprocals of the first three. A study of the first three will be sufficient for our purposes.

$\frac{a}{c}$ is defined to be the sine of $\angle A$.

This is abbreviated as follows: $\sin A = \frac{a}{c}$

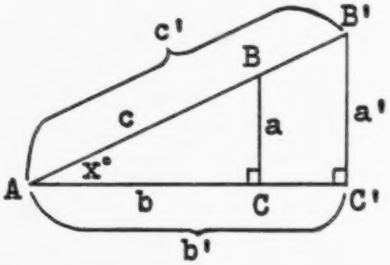
Therefore, for an acute angle, $\angle A$, of a right triangle,

$$\sin A = \frac{\text{length of opposite leg}}{\text{length of hypotenuse}}$$

In $\triangle ABC$, $\sin A = \frac{a}{c}$

In $\triangle AB'C'$, $\sin A = \frac{a'}{c'}$

Since $\triangle ABC \sim \triangle AB'C'$, then $\frac{a}{c} = \frac{a'}{c'}$



From this we conclude that sin A is independent of the lengths of the sides of the triangle. Therefore, if $m\angle A = x$, we may write $\sin x° = \frac{a}{c}$

The same conclusion can be drawn about the other two ratios which are defined and abbreviated as follows.

$\frac{b}{c}$ is the cosine of $\angle A$.

$$\cos A = \frac{b}{c} = \frac{\text{length of adjacent leg}}{\text{length of hypotenuse}}, \quad \cos x^\circ = \frac{b}{c}$$

$\frac{a}{b}$ is the tangent of $\angle A$.

$$\tan A = \frac{a}{b} = \frac{\text{length of opposite leg}}{\text{length of adjacent leg}}, \quad \tan x^\circ = \frac{a}{b}$$

These are all written with respect to acute $\angle A$. Using the same figure, the ratios written with respect to acute $\angle B$ are as follows.

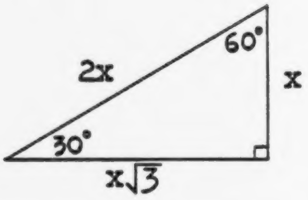
$$\sin B = \frac{b}{c}, \qquad \cos B = \frac{a}{c}, \qquad \tan B = \frac{b}{a}$$

Since the values of these ratios are constant for a specific angle measure, we can determine them for the three special angles of the previous section, 30°, 60°, and 45°. We draw the triangles with the sides expressed in terms of one variable and use the definitions.

$$\sin 30^\circ = \frac{x}{2x} = \frac{1}{2}$$

$$\cos 30^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$$

$$\tan 30^\circ = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$



$$\sin 60^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$$

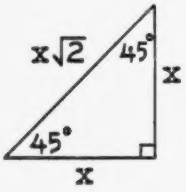
$$\cos 60^\circ = \frac{x}{2x} = \frac{1}{2}$$

$$\tan 60^\circ = \frac{x\sqrt{3}}{x} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\sin 45^\circ = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45^\circ = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45^\circ = \frac{x}{x} = 1$$



If we plan to use these ratios in computation, we would need approximations for the radicals.

The symbol for "is approximately equal to" is $\doteq$.

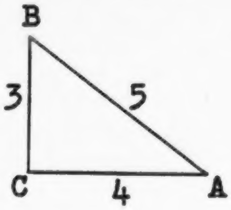
Then $\cos 30^\circ = \frac{\sqrt{3}}{2} \doteq \frac{1.732}{2} \doteq .866$

Fortunately for us, the approximate values of many ratios for angle measures have already been computed to several decimal places. Table on page shows the decimal values correct to four places for all angle measures that are whole numbers from 1 to 89 . This table becomes a powerful tool in computation as shown in the example problems in the exercises.

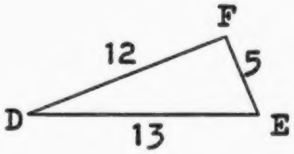
EXERCISE SET 17.7

Each of the following is a right triangle. Complete each statement.

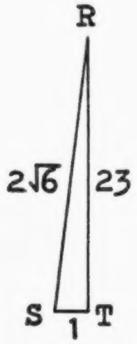
1. sin A =
   cos A =
   tan A =



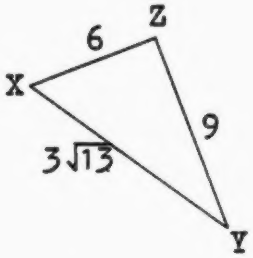
2. sin D =
   cos E =
   sin E =
   cos D =
   tan D =
   tan E =



3. sin R =
   cos S =
   tan R =
   sin S =
   cos R =
   tan S =



4. $\frac{2}{3} =$
   $\frac{2\sqrt{13}}{13} =$
   $\frac{3}{2} =$
   $\frac{3\sqrt{13}}{13} =$



5. If $\sin A = \frac{15}{17}$, find cos A.

6. If $\tan A = \frac{3}{2}$, find sin A.

7. $\sin x^\circ = ?$

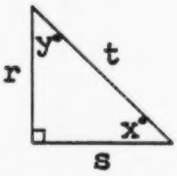
$\cos y^\circ = ?$

$\sin y^\circ = ?$

$\cos x^\circ = ?$

Express y in terms of x.

Complete using x only. $\sin x^\circ = \cos(\ ?\ )^\circ$



8. Can the sine or cosine ratio of an acute angle ever be greater than or equal to 1?

9. When the measure of an angle is close to 0, the sine ratio of that angle is (close to 0, close to 1), and when the measure of an angle is close to 90, the sine ratio of that angle is (close to 0, close to 1).

10. When the measure of an angle is close to 0, the cosine ratio of that angle is (close to 0, close to 1), and when the measure of an angle is close to 90, the cosine ratio of that angle is (close to 0, close to 1).

11. If $m\angle A > 45$, then ($\tan A > 1$, $\tan A < 1$), and if $m\angle A < 45$, then ($\tan A > 1$, $\tan A < 1$).

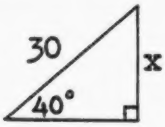
12. Complete the following table using exact values.

	30°	45°	60°
sin			
cos			
tan			

Using $\sqrt{2} \doteq 1.414$ and $\sqrt{3} \doteq 1.732$, verify the table values on page   for these ratios.

When sufficient information about a triangle is given, we can use the table values to find approximations for lengths or angle measures. Answers that are lengths will be rounded off to the same number of digits as the given lengths, and angle measures will be expressed to the nearest degree.

Example 1: Find x.

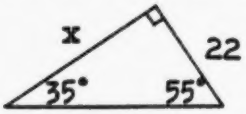


From the table we find $\sin 40^\circ \doteq .6428$

$$\therefore \frac{x}{30} = \sin 40^\circ \longrightarrow \frac{x}{30} \doteq .6428 \longrightarrow x \doteq 30(.6428) \doteq 19$$

Sometimes there is a choice which may simplify the computation.

Example 2: Find x.



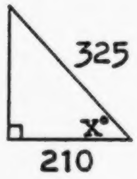
$$\tan 35^\circ = \frac{22}{x} \rightarrow .7002 \doteq \frac{22}{x} \rightarrow .7002x \doteq 22 \rightarrow x \doteq \frac{22}{.7002} \doteq 31$$

OR

$$\frac{x}{22} = \tan 55^\circ \rightarrow \frac{x}{22} \doteq 1.4281 \rightarrow x \doteq 22(1.4281) \doteq 31$$

Which computation would you prefer?

Example 3: Find x.



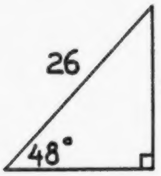
$$\cos x^\circ = \frac{210}{325} \rightarrow \cos x^\circ \doteq .6461$$

From the table we see $\cos 49^\circ \doteq .6561$
$\cos 50^\circ \doteq .6428$

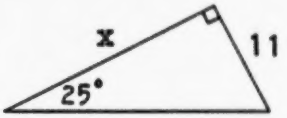
$\therefore x \doteq 50$, since .6461 is closer to .6428 than .6561

Round off all lengths to the same number of digits as the given lengths. Express angle measures to the nearest degree.

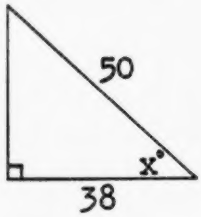
13. Find x.



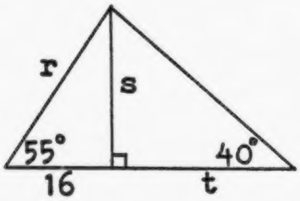
14. Find x.



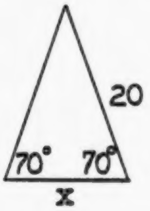
15. Find x.



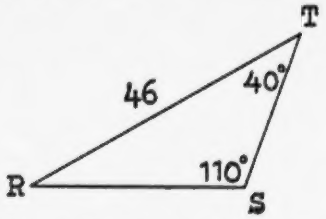
16. Find r, s, and t.



17. Find x.

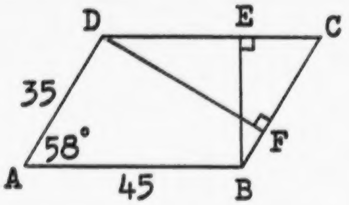


18. Find the length of the altitude from T.



If 46 is the exact length of RT, is your answer exact or approximate?

19. ABCD is a parallelogram. Find BE and DF.



20. A wire 40 feet long is stretched from level ground to the top of a pole 25 feet high. Find the angle the wire makes with the ground.

21. From the top of a mountain that is 4000 feet above a lake, the angle that the line of sight to the edge of the lake makes with the horizontal is 20°. Find the direct distance from the mountain top to the edge of the lake.

22. How high does an airplane rise in flying 5000 feet on a straight path along a line that is 22° above the horizontal?

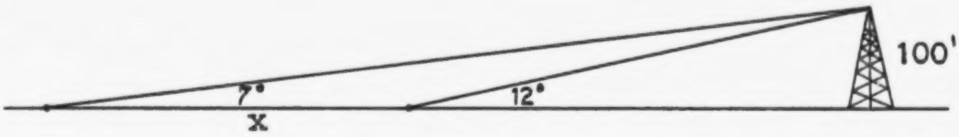
23. From an airplane that is 6180 feet above the ground, the angle between the line of sight to the landing field and the horizontal is 18°. Find the airline distance from the plane to the field.

24. At a point 150 feet from the base of a sequoia tree, the angle between the line of sight to the top of the tree and the level ground is 58°. How high is the tree?

25. An observer on a cliff 220 feet above the water flashes a light to a seaman on board a ship who is 20 feet above the water. The angle between the seaman's line of sight to the light and the horizontal is 6°. Find the distance from the ship to the top of the cliff.

26. A window washer uses a 35 foot ladder. The angle that his ladder makes with the ground must not be more than 75°. Find the shortest distance from the building that he can place the foot of the ladder.

27. At a particular instant that a car is driving toward the base of a 100 foot tower, the angle between the line of sight to the top of the tower and the ground is 7°. At a later point, this angle is 12°. Find the distance the car traveled between the two points.



CHAPTER 18 - CIRCLES AND SPHERES
or LET'S GO ROUND TOGETHER

Definitions

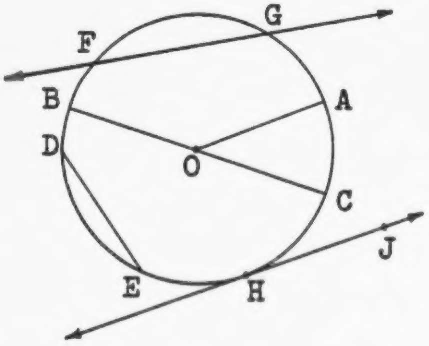
Another of the commonly occurring figures of geometry is the circle. You are already familiar with many of the definitions and some of the properties related to circles. We will state these definitions formally and then prove theorems about circles. Some of the theorems will be easily acceptable by intuition, but some will have interesting conclusions not at all suggested by intuition.

A *circle* is the set of all points in a plane that are at a given distance from a given point in the plane.

The given point is the *center* of the circle. A circle is named by its center. The circle with center P is ⊙P.

The following definitions are illustrated in ⊙O shown below.

A *radius* of a circle is a segment from the center to any point of the circle. $\overline{OA}$ is a radius of ⊙O. The plural of radius is *radii*. $\overline{OA}$ and $\overline{OC}$ are radii of ⊙O. A is the *outer endpoint* of radius $\overline{OA}$.



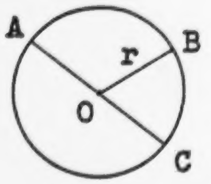
A *chord* of a circle is a segment whose endpoints are points of the circle. $\overline{DE}$ is a chord.

A *diameter* of a circle is a chord that contains the center. $\overline{BC}$ is a diameter.

A *secant* of a circle is a line that contains a chord. $\overleftrightarrow{FG}$ is a secant.

A *tangent* to a circle is a line in the plane of the circle that intersects the circle in exactly one point. If H is the one and only one point on both $\overleftrightarrow{HJ}$ and ⊙O, then $\overleftrightarrow{HJ}$ is tangent to ⊙O at H. H is called the point of tangency.

The words radius and diameter are both used in two ways. The given distance in the definition of a circle is the length of a radius. It is also called "the radius". That is, a circle has an infinite number of "radii" as segments, but only one "radius" as a distance. No confusion should result from this dual usage. For the figure shown, $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ are radii of $\odot$O. The radius is r, where $r = OA = OB = OC$. The same statements hold for the use of the word diameter.



Since all radii of a circle have the same length, we can easily deduce the following statement which you may later use as a reason in proof.

Radii of a circle are congruent.

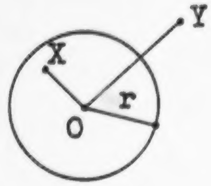
We define congruent circles to be circles with congruent radii.

$\therefore$ Radii of congruent circles are congruent.

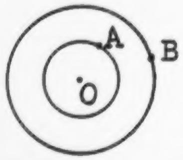
A circle separates a plane into three sets of points: the circle, its interior, and its exterior.

The interior of $\odot$O is the set of all points X, in the plane, such that $OX < r$.

The interior of $\odot$O is the set of all points Y, in the plane, such that $OY > r$.



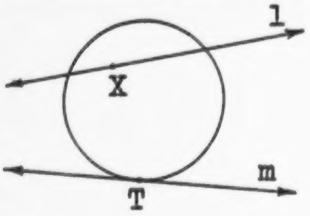
Two or more circles are concentric circles if they are coplanar and have the same center. We would name the two concentric circles shown as $\odot$(O,A) and $\odot$(O,B).



Our first theorem is an obvious statement with a difficult proof. We will accept it at this point and leave the proof as a "hard" exercise.

Theorem 18-1: If a line in the plane of a circle contains an interior point of the circle, then it intersects the circle in exactly two points.

If X is an interior point and l contains X, then l intersects the circle in exactly two points.

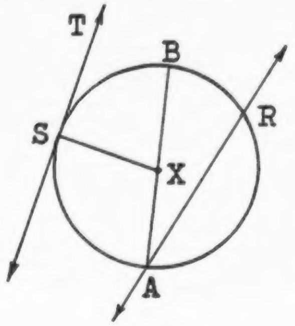


An important consequence of this theorem concerns a tangent. If m is tangent to the circle at T, then all points of m except T must lie in the exterior of the circle. Otherwise, m would intersect the circle in two points which contradicts the definition of a tangent.

EXERCISE SET 18.1

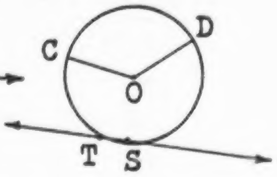
Given $\odot X$. A, B, R, and S are on the circle.

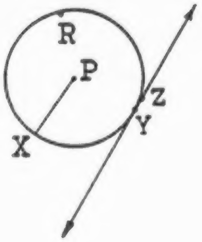
1. Name a radius.
2. Name a diameter.
3. Name a tangent.
4. Name a secant.
5. Name a chord that is not a diameter.



Complete each implication.
C and D are on $\odot O$. X and Y are on $\odot P$.

6. $\overline{OD} \cong \overline{PX} \longrightarrow$
7. $\left.\begin{array}{l}\overline{OC} \text{ is a radius of } \odot O\\ \overline{OD} \text{ is a radius of } \odot O\end{array}\right\} \longrightarrow$
8. $\odot O \cong \odot P \longrightarrow$





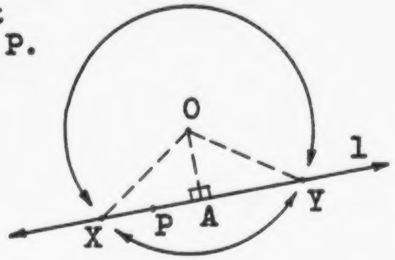
9. R is in the interior of $\odot P \longrightarrow PR$ _?_ PX
10. T is in the exterior of $\odot O \longrightarrow OT$ _?_ OC
11. S is in the interior of $\odot O \longrightarrow \overleftrightarrow{ST}$ intersects $\odot O$ in _?_ points
12. $\overleftrightarrow{YZ}$ is tangent to $\odot P$ at Y $\longrightarrow$ Z is (in the interior of $\odot P$, on $\odot P$, in the exterior of $\odot P$)
13. Construct a circle and call the center O. At a point, A, on $\odot O$ draw a line, l, that appears to be tangent to $\odot O$. Draw $\overline{OA}$. What relationship appears to exist between l and $\overline{OA}$?
14. Construct a circle and call the center O. Through a point, A, on the circle draw $\overline{OA}$. At A construct a line, l, such that $l \perp \overline{OA}$. What relationship does l appear to have to $\odot O$?

15. Construct a circle and call the center O. At a point, A, on circle O draw a line, l, that appears to be tangent to ⊙O. At A construct a line, m, such that $m \perp l$. What other point besides A does m appear to contain?

16. Complete the following proof of theorem 18-1.

Given: P is an interior point of ⊙O and l contains P. Radius is r.

Prove: l intersects ⊙O in exactly two points



Argument

Let A be the foot of the perpendicular from O to l.

$$\left.\begin{array}{r}\text{P is an interior point of } \odot O \longrightarrow OP < r\\ OA \perp l \longrightarrow OA < OP\end{array}\right\} \longrightarrow OA < r \longrightarrow$$

$$\longrightarrow r - OA \text{ is positive} \longrightarrow r^2 - (OA)^2 \text{ is positive}$$

Let X be a point on l such that $AX = \sqrt{r^2 - (OA)^2}$

$$\left.\begin{array}{l}AX = \sqrt{r^2 - (OA)^2} \longrightarrow (AX)^2 = r^2 - (OA)^2 \longrightarrow (AX)^2 + (OA)^2 = r^2\\ OA \perp l \longrightarrow \angle OAX \text{ is a rt. } \angle \longrightarrow (AX)^2 + (OA)^2 = (OX)^2\end{array}\right\} \longrightarrow$$

$$\longrightarrow (OX)^2 = r^2 \longrightarrow OX = r \longrightarrow x \text{ is on } \odot O$$

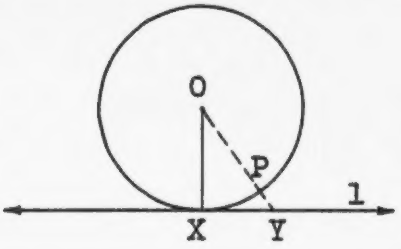
Show that there is another point, Y, on l satisfying the same conditions as X and tell why X and Y are the only two points satisfying the conditions. (Both on l and both on ⊙O)

Theorems About Tangents

Theorem 18-2: A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

Given: l tangent to $\odot O$ at X

Prove: $\overline{OX} \perp l$



Indirect argument

① Suppose $\overline{OX} \not\perp l$.
② Let $\overline{OY}$ be perpendicular to l at Y.
③ Since X is the point of tangency, Y is in the interior of $\odot O$.
④ Let P be the point of intersection of $\overline{OY}$ and $\odot O$.

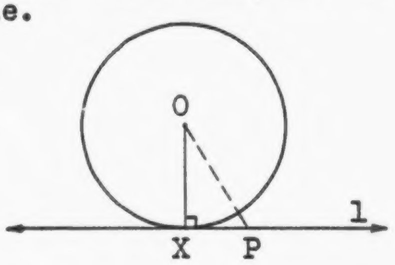
$$\left.\begin{array}{r} \text{⑤}\ OP+PY=OY \xrightarrow{\text{⑥}} \left.\begin{array}{l}OY>OP\\OP=OX\end{array}\right\} \xrightarrow{\text{⑧}} OY>OX \\ \text{②}\ \overline{OY}\perp l \xrightarrow{\text{⑨}} OY<OX\end{array}\right\} \xrightarrow{\text{⑩}} \begin{array}{l}\overline{OX}\not\perp l \text{ is false}\\ \text{and } \overline{OX}\perp l \text{ is true}\end{array}$$

∴ l tangent to $\odot O$ at X $\longrightarrow \overline{OX} \perp l$

Theorem 18-3: A line in the plane of a circle and perpendicular to a radius at its outer endpoint is tangent to the circle.

Given: In $\odot O$, $l \perp$ radius $\overline{OX}$ at X

Prove: l is tangent to $\odot O$

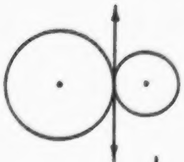


Argument

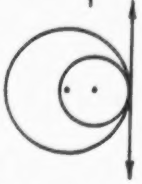
① Let P be any other point on l.
② Draw $\overline{OP}$
③ $\overline{OX} \perp l \xrightarrow{\text{④}} OP > OX \xrightarrow{\text{⑤}}$ P is in the exterior of the circle

∴ All other points of l except X are in the exterior of the circle and l is tangent to the circle.

Two circles are *tangent circles* if and only if they are coplanar and are tangent to the same line at the same point.



If the centers lie on opposite sides of the common tangent, the circles are *externally tangent*.



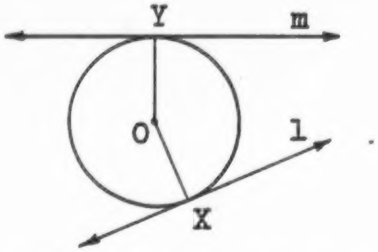
If the centers lie on the same side of the common tangent, the circles are *internally tangent*.

Of course, this definition requires that tangent circles intersect in exactly one point.

EXERCISE SET 18.2

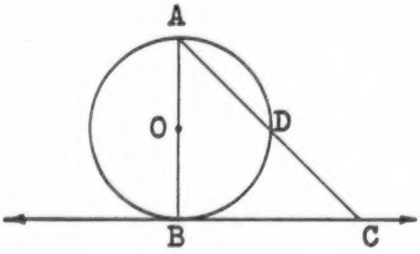
Complete each implication.

1. l tangent to $\odot O$ at $X \longrightarrow$
2. $\left.\begin{array}{l} m \perp \overline{OY} \text{ at } Y \\ Y \text{ is on } \odot O \end{array}\right\} \longrightarrow$

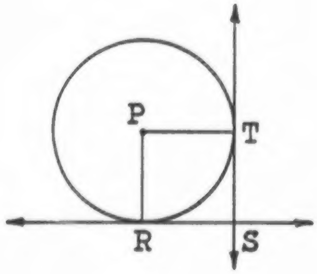


3. Prove: If two different lines are both tangent to a circle at the endpoints of the same diameter, they are parallel.

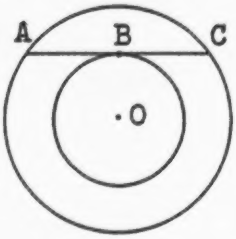
*4. Given: $\overline{BC}$ tangent to $\odot O$ at B. $\overline{BC} \cong \overline{BA}$
Prove: $\overline{AD} \cong \overline{DC}$



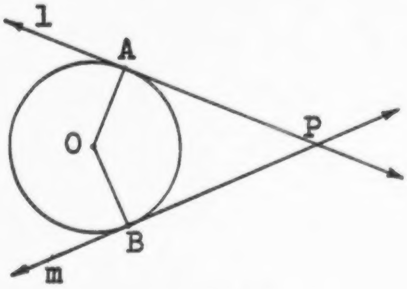
5. Given: R and T are on $\odot P$
$\overline{ST}$ tangent to $\odot P$
$\overline{PR} \cong \overline{TS}$, $\overline{PR} \parallel \overline{TS}$
Prove: $\overline{RS}$ tangent $\odot P$



6. Given: $\odot(O,A)$ and $\odot(O,B)$
are concentric circles
$\overleftrightarrow{AC}$ tangent to $\odot O$ at B
C is on $\odot(O,A)$
Prove: $\overline{AB} \cong \overline{CB}$
(Hint: draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$)

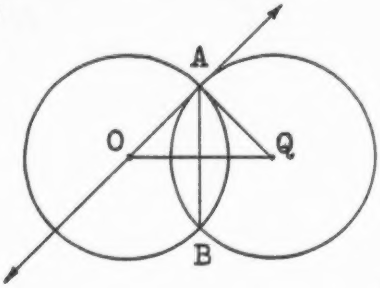


7. Given: l tangent to $\odot O$ at A
m tangent to $\odot O$ at B
l and m intersect at P
Prove: $\angle APB$ suppl. $\angle AOB$

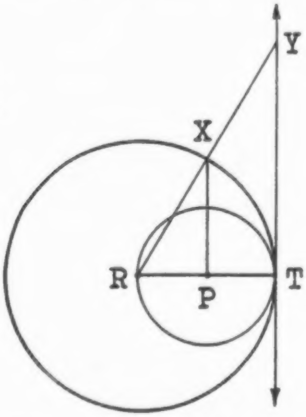


8. Prove: If two circles are tangent circles, the two centers and the point of tangency are collinear.

9. Given: $\odot O$ and $\odot Q$ intersect
at A and B. $\odot O \cong \odot Q$
$\overline{OA}$ tangent to $\odot Q$
Prove: $\overline{OQ} \cong \overline{AB}$



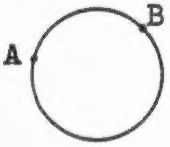
10. Given: $\odot R$ and $\odot P$, $\odot P$ contains R
$\overleftrightarrow{TY}$ tangent to both $\odot$s at T
$\overline{PX} \parallel \overleftrightarrow{TY}$
Prove: $TY = (RT)\sqrt{3}$



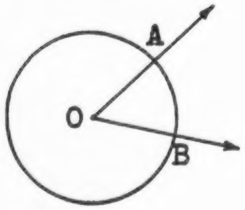
11. Prove: A line perpendicular to a tangent of a circle at the point of tangency contains the center of the circle.

Arcs of Circles

Two points of a circle will separate the circle into two parts, each called an *arc* of the circle. Each arc contains the two given points as its endpoints.

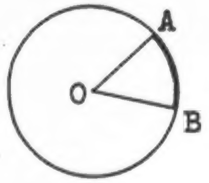


A *central angle* of a circle is an angle whose vertex is the center of the circle. $\angle AOB$ is a central angle of $\odot O$.

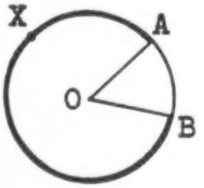


Every central angle will intersect a circle in two points determining two arcs which we now define.

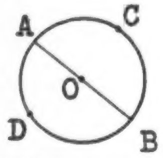
A *minor arc* of a circle is the union of two points of the circle, not the endpoints of a diameter, and all points of the circle that lie in the interior of the central angle whose sides contain the points. The minor arc in this figure is named $\overset{\frown}{AB}$.



A *major arc* of a circle is the union of two points of the circle, not the endpoints of a diameter, and all points of the circle that lie in the exterior of the central angle whose sides contain the points. Since the endpoints of the major arc are the same as the endpoints of a minor arc, we use three letters to name a major arc. $\overset{\frown}{AXB}$ is a major arc of $\odot O$.



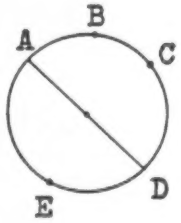
A *semicircle* is the union of the endpoints of a diameter and all points of the circle on one side of the diameter. $\overset{\frown}{ACB}$ and $\overset{\frown}{ADB}$ are semicircles of $\odot O$.



Keep the following points in mind when naming arcs.

Two points *always* name a minor arc.
A minor arc *may be* named with three points.
A major arc *must be* named with three points.
A semicircle *must be* named with three points.

$\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{ABC}$, $\overset{\frown}{BCD}$ all name minor arcs.
$\overset{\frown}{ACE}$, $\overset{\frown}{BDA}$, $\overset{\frown}{CAD}$ all name major arcs.
$\overset{\frown}{ABD}$, $\overset{\frown}{AED}$, $\overset{\frown}{DCA}$ all name semicircles.



For any minor arc of a circle, there is a corresponding major arc, such that every point of the circle belongs to one or the other with only the endpoints belonging to both. We now define the degree measure of arcs but we will refer to it simply as the measure. The measure of $\overset{\frown}{AB}$ is written $m\overset{\frown}{AB}$.

The measure of a minor arc is the measure of its central angle.

The measure of a semicircle is 180.

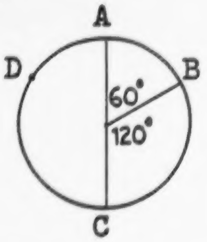
The measure of a major arc is 360 minus the measure of the corresponding minor arc.

$m\overset{\frown}{AB} = 60$, $m\overset{\frown}{BC} = 120$

$m\overset{\frown}{ABC} = 180$, $m\overset{\frown}{ADC} = 180$

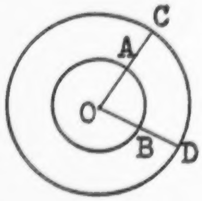
$m\overset{\frown}{ACB} = 360 - 60 = 300$

$m\overset{\frown}{BAC} = 360 - 120 = 240$



Notice that the degree measure of an arc does not depend upon the size of the circle containing it.

$m\overset{\frown}{AB} = m\overset{\frown}{CD}$

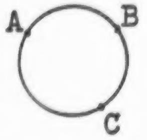


The statements involving addition of arc measures are similar in structure to statements involving addition of segment lengths and angle measures. But there is no betweenness of points on a circle because no three points of a circle are collinear.

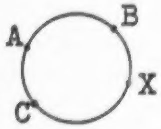
The following postulate has two cases to account for the different combinations of arcs whose measures are to be added.

Postulate 24: Arc Addition Postulate

Case I: If the intersection of minor arc $\overset{\frown}{AB}$ and minor arc $\overset{\frown}{BC}$ is the single point B, then $m\overset{\frown}{AB} + m\overset{\frown}{BC} = m\overset{\frown}{ABC}$. ($\overset{\frown}{ABC}$ may be a minor arc, semicircle, or major arc.)



Case II: If $\overset{\frown}{AB}$ is a minor arc and $\overset{\frown}{BXC}$ is a semicircle or major arc, and the intersection is the single point B, then $m\overset{\frown}{AB} + m\overset{\frown}{BXC} = m\overset{\frown}{ABC}$.



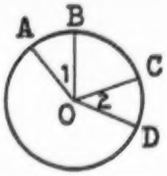
In the same circle or congruent circles, two arcs are congruent arcs if and only if their measures are equal.

$m\widehat{AB} = m\widehat{CD} \longleftrightarrow \widehat{AB} \cong \widehat{CD}$

Theorem 18-4: In the same circle or congruent circles, if two central angles are congruent, their arcs are congruent.

Given: $\angle 1 \cong \angle 2$

Prove: $\widehat{AB} \cong \widehat{CD}$



Argument

$$\left.\begin{array}{l} \textcircled{1}\ m\widehat{AB} = m\angle 1 \\ \textcircled{2}\ m\widehat{CD} = m\angle 2 \\ \quad \textcircled{3}\ \angle 1 \cong \angle 2 \end{array}\right\} \longrightarrow \cdots \xrightarrow{\textcircled{4}} \widehat{AB} \cong \widehat{CD}$$

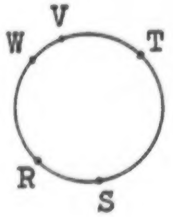
$\therefore \angle 1 \cong \angle 2 \longrightarrow \widehat{AB} \cong \widehat{CD}$

Theorem 18-5: In the same circle or congruent circles, if two arcs are congruent, then their central angles are congruent.

The proof is left as an exercise.

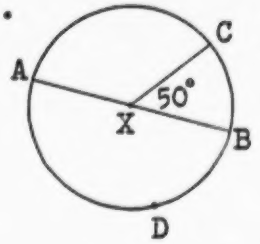
EXERCISE SET 18.3

R and T are endpoints of a diameter.

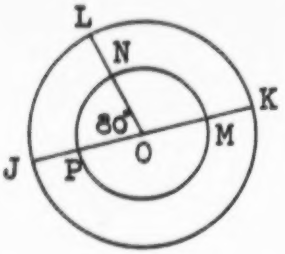


1. Name a minor arc with endpoints W and S.
2. Name a semicircle with endpoints R and T. and containing S.
3. Name a major arc with endpoints T and S.
4. Name a major arc with endpoints V and W.
5. If V and S are not the endpoints of a diameter, $\widehat{VS}$ names a _____.
6. Is V a point of $\widehat{WT}$?
7. Is R a point of $\widehat{TSW}$?
8. Is W a point of $\widehat{VTR}$?
9. $m\widehat{RW} + m\widehat{WV} =$ _____
10. $m\widehat{TS} + m\widehat{SR} =$ _____
11. $m\widehat{SW} + m\widehat{WV} =$ _____
12. $m\widehat{WT} + m\widehat{TS} =$ _____
13. $m\widehat{WRT} + m\widehat{TV} =$ _____

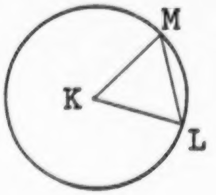
14. Name two central angles of ⊙X.
15. $m\widehat{BC} =$ _____
16. $m\widehat{ADB} =$ _____
17. $m\widehat{AC} =$ _____
18. $m\widehat{CDB} =$ _____
19. $m\widehat{ABC} =$ _____



20. Which has the greater measure, $\widehat{JL}$ or $\widehat{MN}$?

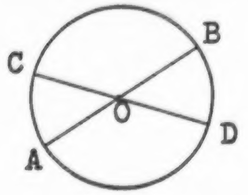


21. Given: ⊙K, $m\widehat{LM} = 60$
Prove: ΔKLM is equilateral

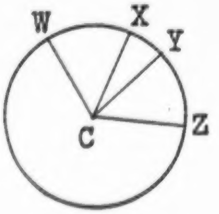


22. Prove theorem 18-5.

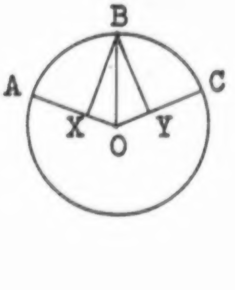
23. Given: $\overline{AB}$ and $\overline{CD}$ are diameters of ⊙O
Prove: $\widehat{AC} \cong \widehat{BD}$



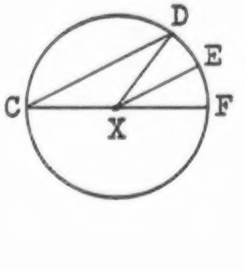
24. Given: ⊙C, $\widehat{WY} \cong \widehat{XZ}$
Prove: ∠WCX ≅ ∠YCZ



25. Given: ⊙O, $\overline{BX} \perp \overline{OA}$
$\overline{BY} \perp \overline{OC}$, BX ≅ BY
Prove: $\widehat{AB} \cong \widehat{CB}$

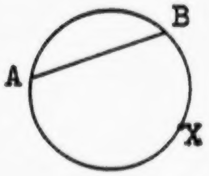


26. Given: ⊙X, $\overline{CD} \parallel \overline{XE}$
Prove: $\widehat{DE} \cong \widehat{EF}$



Arcs and Related Chords

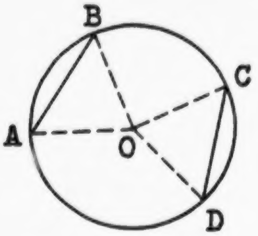
For any chord of a circle, there will be two related arcs. If the chord is not a diameter, the arcs will be a major arc and a minor arc. We will consider the arc of a chord to be the minor arc. That is, the arc of $\overline{AB}$ is $\overset{\frown}{AB}$.



Theorem 18-6: In the same circle or congruent circles, congruent chords have congruent arcs.

Given: $\overline{AB} \cong \overline{CD}$ in $\odot O$

Prove: $\overset{\frown}{AB} \cong \overset{\frown}{CD}$



Argument

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① Draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$.

$$\left.\begin{array}{l} \text{②}\ \overline{OA} \cong \overline{OC} \\ \text{③}\ \overline{OB} \cong \overline{OD} \\ \text{④}\ \overline{AB} \cong \overline{CD} \end{array}\right\} \xrightarrow{\text{⑤}} \triangle AOB \cong \triangle COD \xrightarrow{\text{⑥}} \angle AOB \cong \angle COD \xrightarrow{\text{⑦}} \overset{\frown}{AB} \cong \overset{\frown}{CD}$$

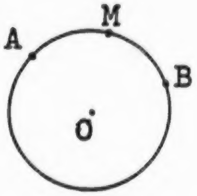
$$\therefore\ \overline{AB} \cong \overline{CD} \longrightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$$

Theorem 18-7: In the same circle or congruent circles, congruent arcs have congruent chords.

The proof is left as an exercise.

The midpoint of an arc is the point that divides it into two congruent arcs.

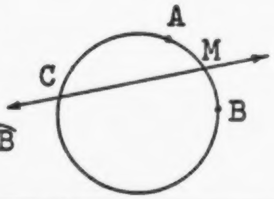
M is the midpoint of $\overset{\frown}{AB} \longleftrightarrow \overset{\frown}{AM} \cong \overset{\frown}{MB}$



Do not confuse the midpoint of an arc with the center of the arc which is the center of the circle containing the arc. The center of $\overset{\frown}{AB}$ is O.

A bisector of an arc is a line, ray, or segment that intersects the arc at its midpoint.

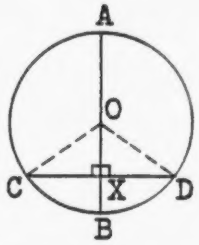
M is midpoint of $\overset{\frown}{AB} \longleftrightarrow \overrightarrow{CM}$ bisects $\overset{\frown}{AB}$



Theorem 18-8: A diameter that is perpendicular to a chord bisects the chord and its two arcs.

Given: $\overline{AB}$ is a diameter of $\odot O$
$\overline{AB}\perp\overline{CD}$

Prove: $\overline{CX}\cong\overline{DX}$, $\overset{\frown}{CB}\cong\overset{\frown}{DB}$, $\overset{\frown}{CA}\cong\overset{\frown}{DA}$



Argument

① Draw $\overline{OC}$ and $\overline{OD}$

$$\left.\begin{array}{r} \text{②}\ \overline{OX}\perp\overline{CD} \longrightarrow \dots \xrightarrow{\text{③}} \triangle OXC \text{ is a rt. } \triangle \\ \searrow \dots \xrightarrow{\text{④}} \triangle OXD \text{ is a rt. } \triangle \\ \text{⑤}\ \overline{OC}\cong\overline{OD} \\ \text{⑥}\ \overline{OX}\cong\overline{OX} \end{array}\right\} \xrightarrow{\text{⑦}} \triangle OXC \cong \triangle OXD \begin{array}{l} \xrightarrow{\text{⑧}} \\ \xrightarrow{\text{⑨}} \end{array}$$

$$\begin{array}{l} \xrightarrow{\text{⑧}} \overline{CX}\cong\overline{DX} \\ \xrightarrow{\text{⑨}} \angle COB \cong \angle DOB \xrightarrow{\text{⑩}} \overset{\frown}{CB}\cong\overset{\frown}{DB} \end{array}$$

$$\left.\begin{array}{r} \text{⑪}\ \angle AOC \text{ suppl. } \angle COB \\ \text{⑫}\ \angle AOD \text{ suppl. } \angle DOB \\ \text{⑨}\ \angle COB \cong \angle DOB \end{array}\right\} \xrightarrow{\text{⑬}} \angle AOC \cong \angle AOD \xrightarrow{\text{⑭}} \overset{\frown}{CA}\cong\overset{\frown}{DA}$$

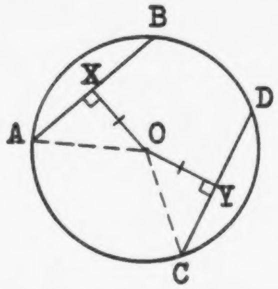
$$\therefore \left.\begin{array}{r} \overline{AB} \text{ is a diameter of } \odot O \\ \overline{AB}\perp\overline{CD} \end{array}\right\} \begin{array}{l} \nearrow \overline{CX}\cong\overline{DX} \\ \rightarrow \overset{\frown}{CB}\cong\overset{\frown}{DB} \\ \searrow \overset{\frown}{CA}\cong\overset{\frown}{DA} \end{array}$$

The distance from a point to a line has been defined as the length of the perpendicular segment from the point to the line. If a theorem refers to the distance from a point to a line, the perpendicular relation is considered as given as in the following theorem.

Theorem 18-9: In the same circle or in congruent circles, chords that are equally distant from the center are congruent.

Given: In $\odot O$, $\overline{OX} \perp \overline{AB}$, $\overline{OY} \perp \overline{CD}$
$OX = OY$

Prove: $\overline{AB} \cong \overline{CD}$



Argument

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① Draw $\overline{OA}$ and $\overline{OC}$

$$\left.\begin{array}{r} \text{②}\ \overline{OX} \perp \overline{AB} \rightarrow \dots \xrightarrow{\text{③}} \triangle OXA \text{ is a rt. } \triangle \\ \text{④}\ \overline{OY} \perp \overline{CD} \rightarrow \dots \xrightarrow{\text{⑤}} \triangle OYC \text{ is a rt. } \triangle \\ \text{⑥}\ \overline{OA} \cong \overline{OC} \\ \text{⑦}\ \overline{OX} \cong \overline{OY} \end{array}\right\} \xrightarrow{\text{⑧}} \triangle OXA \cong \triangle OYC \xrightarrow{\text{⑨}} \overline{AX} \cong \overline{CY}$$

$$\left.\begin{array}{r} \text{②}\ \overline{OX} \perp \overline{AB} \xrightarrow{\text{⑩}} \overline{AX} \cong \overline{BX} \xrightarrow{\text{⑪}} AX = \frac{1}{2} \cdot AB \\ \text{④}\ \overline{OY} \perp \overline{CD} \xrightarrow{\text{⑫}} \overline{CY} \cong \overline{DY} \xrightarrow{\text{⑬}} CY = \frac{1}{2} \cdot CD \\ \text{⑨}\ AX = CY \end{array}\right\} \xrightarrow{\text{⑭}} \frac{1}{2} AB = \frac{1}{2} CD \xrightarrow{\text{⑮}} \overline{AB} \cong \overline{CD}$$

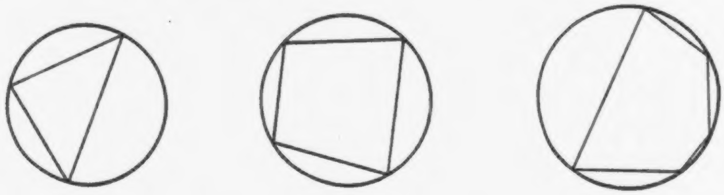
$$\therefore \left.\begin{array}{l} \overline{OX} \perp \overline{AB} \\ \overline{OY} \perp \overline{CD} \\ OX = OY \end{array}\right\} \longrightarrow \overline{AB} \cong \overline{CD}$$

Theorem 18-10: In the same circle or in congruent circles, congruent chords are equally distant from the center.

The proof is left as an exercise.

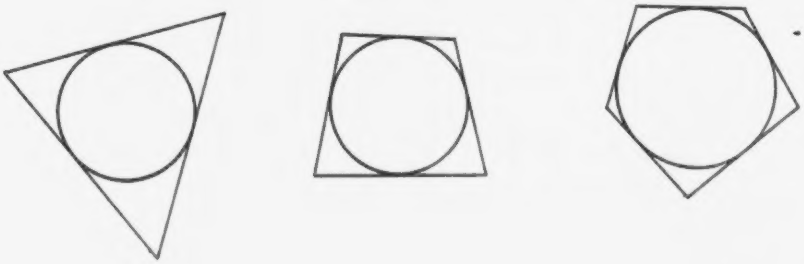
Study carefully the following definitions which state relationships between polygons and circles.

A polygon is *inscribed in a circle* if every vertex of the polygon is a point of the circle. The circle is said to be *circumscribed about the polygon*.



The polygon is inscribed in the circle.
The circle is circumscribed about the polygon.

A polygon is *circumscribed about a circle* if every side of the polygon is tangent to the circle. The circle is said to be *inscribed in the polygon*.



The polygon is circumscribed about the circle.
The circle is inscribed in the polygon.

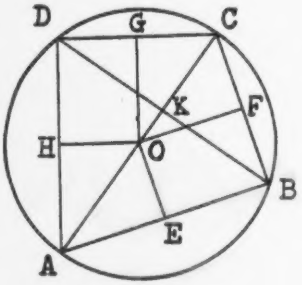
EXERCISE SET 18.4

1. Prove theorem 18-7.
2. Prove theorem 18-10.

Complete each implication about ⊙O.

3. $\left.\begin{array}{l} OG = OF \\ \overline{OG} \perp \overline{DC} \\ \overline{OF} \perp \overline{BC} \end{array}\right\} \longrightarrow$

4. $\left.\begin{array}{r} \overline{AC} \text{ is a diameter} \\ \overline{AC} \perp \overline{BD} \end{array}\right\} \begin{array}{l} \nearrow \\ \rightarrow \\ \searrow \end{array}$

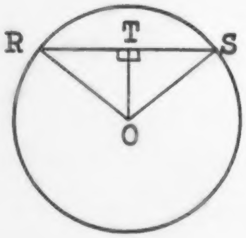


5. $\overset{\frown}{AD} \cong \overset{\frown}{AB} \longrightarrow$

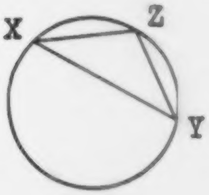
6. $\left.\begin{array}{l} \overline{AD} \cong \overline{AB} \\ \overline{OH} \perp \overline{AD} \\ \overline{OE} \perp \overline{AB} \end{array}\right\} \longrightarrow$

7. $\overline{DC} \cong \overline{CB} \longrightarrow$

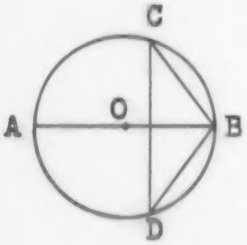
Given ⊙O with $\overline{OT} \perp \overline{RS}$.



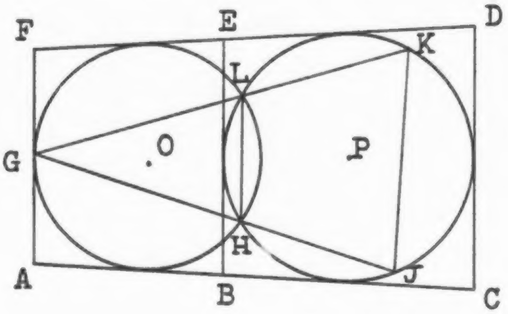
8. If RS = 8, OT = 3, find OR.
9. If OS = 13, OT = 5, find RS.
10. If OR = 6, RS = 10, find OT.
11. If m ROS = 90, OT = 8, find RS.
12. If m RSO = 30, OR = 8, find RS.
13. If a chord of a circle has length 14 and the radius of the circle is 9, what is the distance from the center to the chord?
14. The length of one side of a square inscribed in a circle is 6. Find the radius of the circle.
15. Given: ∠X ≅ ∠Y
    Prove: $\overline{XZ} \cong \overline{YZ}$



16. Given: $\overline{AB}$ is a diameter of ⊙O, $\overline{AB} \perp \overline{CD}$
    Prove: △BCD is isosceles



Use the appearance of the given figure.

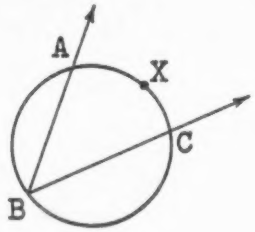


17. Name an inscribed triangle.
18. Name an inscribed circle.
19. Name a circumscribed quadrilateral.
20. Name a circle circumscribed about a triangle.
21. Name an inscribed quadrilateral.

Angles and Intercepted Arcs

The only angle related to a circle that we've discussed so far is a central angle whose vertex is the center of the circle. Now we will consider other angles and their related arcs.

An inscribed angle of a circle is an angle whose vertex is on the circle and whose sides each contain a chord of the circle.



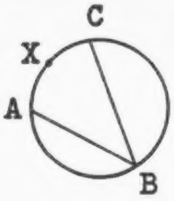
An inscribed angle is inscribed in the arc whose endpoints are the intersections of the sides of the angle with the circle, and that contains the vertex of the angle as an interior point.

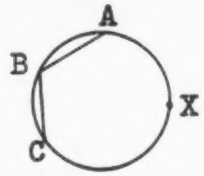
$\angle ABC$ is an inscribed angle and is inscribed in $\widehat{ABC}$.

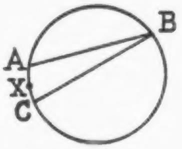
An inscribed angle intercepts an arc whose endpoints are the intersections of the sides of the angle with the circle, and all other points lie in the interior of the angle. (Do not confuse "intercepts" with "intersects".)

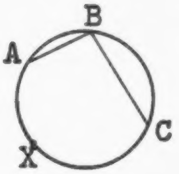
Inscribed angle $\angle ABC$ intercepts $\widehat{AXC}$.

In many problems, the sides of the angle are shown as chords rather than rays. Each of the figures below shows an example of inscribed angle $\angle ABC$, which is inscribed in $\widehat{ABC}$ and intercepts $\widehat{AXC}$.







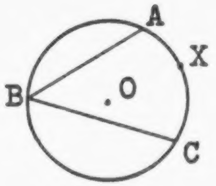


The next theorem about the inscribed angle involves three separate cases, with the first case being used to prove the other two. All three must be proved if the theorem is to be true in general for all inscribed angles.

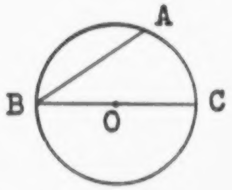
Theorem 18-11: The measure of an inscribed angle of a circle is half the measure of its intercepted arc.

Given: $\angle ABC$ is an inscribed angle of $\odot O$

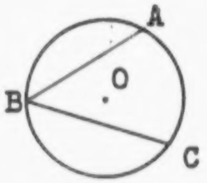
Prove: $m\angle ABC = \frac{1}{2} m\widehat{AXC}$



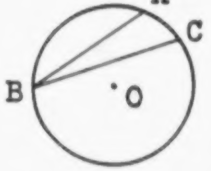
The best plan is to first prove it for the case in which one side of the angle is a diameter, and then consider the cases where the center is in the interior of the angle and the exterior of the angle.



Case I



Case II

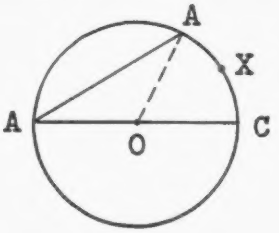


Case III

Case I

Given: $\overline{BC}$ is a diameter of $\odot O$

Prove: $m\angle ABC = \frac{1}{2}\, m\, \widehat{AXC}$



Argument

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① Draw $\overline{AO}$

$$\text{②}\ \left.\begin{array}{r} \overline{AO} \cong \overline{BO} \xrightarrow{\text{③}} m\angle BAO = m\angle ABO \\ \text{④}\ m\angle BAO + m\angle ABO = m\angle AOC \end{array}\right\} \xrightarrow{\text{⑤}} m\angle ABO + m\angle ABO = m\angle AOC \xrightarrow{\text{⑥}}$$

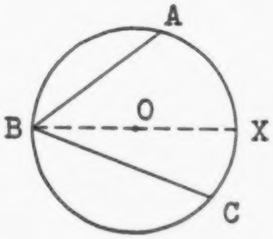
$$\xrightarrow{\text{⑥}} 2\cdot m\angle ABO = m\angle AOC \xrightarrow{\text{⑦}} \left.\begin{array}{r} m\angle ABO = \frac{1}{2} m\angle AOC \\ \text{⑧}\ m\angle AOC = m\,\widehat{AXC} \end{array}\right\} \xrightarrow{\text{⑨}} m\angle ABC = \frac{1}{2} m\,\widehat{AXC}$$

$$\therefore \left.\begin{array}{r} \angle ABC \text{ is an inscribed } \angle \text{ of } \odot O \\ \overline{BC} \text{ is a diameter of } \odot O \end{array}\right\} \longrightarrow m\angle ABC = \frac{1}{2} m\,\widehat{AXC}$$

Case II

Given: The center O lies in the interior of inscribed $\angle ABC$

Prove: $m\angle ABC = \frac{1}{2} m\,\widehat{AXC}$



Argument

---

① Draw diameter $\overline{BX}$

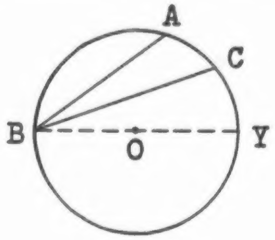
$$\left.\begin{array}{l} \text{②}\ m\angle ABX = \frac{1}{2} m\,\widehat{AX} \\ \text{③}\ m\angle XBC = \frac{1}{2} m\,\widehat{XC} \end{array}\right\} \xrightarrow{\text{④}} m\angle ABX + m\angle XBC = \frac{1}{2} m\,\widehat{AX} + \frac{1}{2} m\,\widehat{XC} \xrightarrow{\text{⑤}}$$

$$\left.\begin{array}{r} \xrightarrow{\text{⑤}} m\angle ABX + m\angle XBC = \frac{1}{2}(m\widehat{AX} + m\widehat{XC}) \\ \text{⑥}\ m\angle ABX + m\angle XBC = m\angle ABC \\ \text{⑦}\ m\widehat{AX} + m\widehat{XC} = m\widehat{AXC} \end{array}\right\} \xrightarrow{\text{⑧}} m\angle ABC = \frac{1}{2} m\,\widehat{AXC}$$

$$\therefore O \text{ is in interior of inscribed } \angle ABC \longrightarrow m\angle ABC = \frac{1}{2} m\,\widehat{AXC}$$

Case III

The argument for case III is similar to that for case II. We will show just steps ②, ③, and ④.



$$\left.\begin{array}{l} \text{②}\ m\angle ABY = \frac{1}{2} m\,\widehat{AY} \\ \text{③}\ m\angle CBY = \frac{1}{2} m\,\widehat{CY} \end{array}\right\} \xrightarrow{\text{④}} m\angle ABY - m\angle CBY = \frac{1}{2} m\,\widehat{AY} - \frac{1}{2} m\,\widehat{CY} \xrightarrow{\text{⑤}} \ldots$$

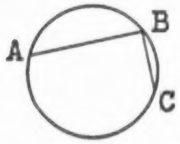
You may complete the proof.

∴ If $\angle ABC$ is any inscribed angle which intercepts $\widehat{AXC}$, then $m\angle ABC = \frac{1}{2} m\,\widehat{AXC}$.

Theorem 18-11 has four corollaries which are step savers in many proofs. The first three will be left as exercises.

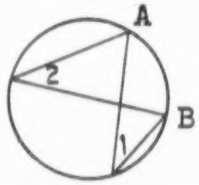
Corollary 18-11a: An angle inscribed in a semicircle is a right angle.

$\widehat{ABC}$ is a semicircle $\longrightarrow \angle ABC$ is a rt. $\angle$



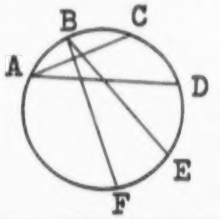
Corollary 18-11b: If two inscribed angles intercept the same arc, they are congruent.

$$\left.\begin{array}{l} \angle 1 \text{ intercepts } \widehat{AB} \\ \angle 2 \text{ intercepts } \widehat{AB} \end{array}\right\} \longrightarrow \angle 1 \cong \angle 2$$



Corollary 18-11c: If two inscribed angles intercept congruent arcs, they are congruent.

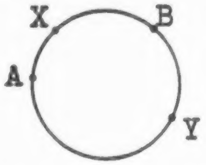
$\widehat{CD} \cong \widehat{EF} \longrightarrow \angle A \cong \angle B$



Before considering the last corollary, the following is a statement you may use in a proof.

The sum of the measures of a minor arc and its related major arc is 360.

$m\,\widehat{AXB} + m\,\widehat{AYB} = 360$

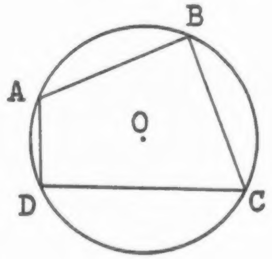


The same is obviously true for the sum of the measures of two semicircles.

Corollary 18-11d: Opposite angles of an inscribed quadrilateral are supplementary.

Given: ABCD is inscribed in ⊙O

Prove: ∠A suppl. ∠C
∠B suppl. ∠D



Argument

$$\left.\begin{array}{l} \textcircled{1}\ m\angle A = \frac{1}{2} m\,\widehat{BCD} \\ \textcircled{2}\ m\angle C = \frac{1}{2} m\,\widehat{BAD} \end{array}\right\} \xrightarrow{\textcircled{3}} m\angle A + m\angle C = \frac{1}{2} m\,\widehat{BCD} + \frac{1}{2} m\,\widehat{BAD} \xrightarrow{\textcircled{4}}$$

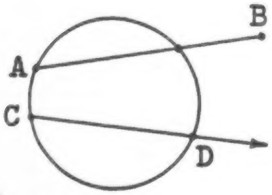
$$\left.\begin{array}{r} \xrightarrow{\textcircled{4}} m\angle A + m\angle C = \frac{1}{2}(m\,\widehat{BCD} + m\,\widehat{BAD}) \\ \textcircled{5}\ m\,\widehat{BCD} + m\,\widehat{BAD} = 360 \end{array}\right\} \xrightarrow{\textcircled{6}} m\angle A + m\angle C = \frac{1}{2}(360) = 180 \xrightarrow{\textcircled{7}}$$

$\xrightarrow{\textcircled{7}}$ ∠A suppl. ∠C

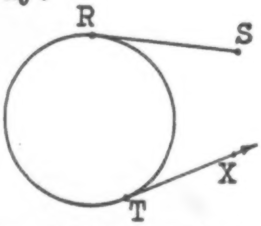
Similarly, ∠B suppl. ∠D

$$\therefore \text{ABCD is inscribed in } \odot O \begin{array}{l} \nearrow \angle A \text{ suppl. } \angle C \\ \searrow \angle B \text{ suppl. } \angle D \end{array}$$

A secant and a tangent have both been defined as lines. However, many figures will show them as rays or segments. A ray is a secant ray if it contains a chord. A segment is a secant segment if it contains a chord and both endpoints are not on the circle. $\overline{AB}$ is a secant segment. $\overrightarrow{CD}$ is a secant ray.



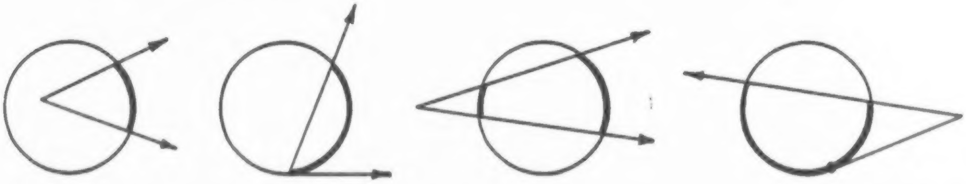
A ray or segment is a tangent ray or tangent segment if the line determined is a tangent line.
$\overline{RS}$ is a tangent segment.
$\overrightarrow{TX}$ is a tangent ray.



For angles other than inscribed angles, intercepted arcs are similarly defined.

An angle intercepts an arc or arcs of a circle if

1. The endpoints of the arc lie on the sides of the angle with at least one endpoint on each side.
2. All other points of the arc lie in the interior of the angle.



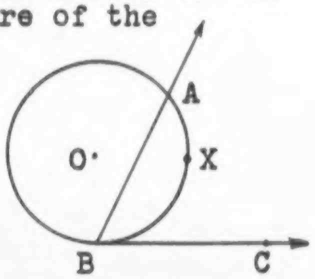
In the two left figures the angle has one intercepted arc. In the two right figures the angle has two intercepted arcs.

The proof of the next theorem has three cases similar to the proof of theorem 18-11. We will show the proof for case I and illustrate it for cases II and III.

Theorem 18-12: The measure of an angle formed by a secant ray and a tangent ray with vertex on the circle is half the measure of the intercepted arc.

Given: $\overrightarrow{BC}$ is tangent to $\odot O$ at B
$\angle ABC$ intercepts $\widehat{AXB}$

Prove: $m\angle ABC = \frac{1}{2} m\,\widehat{AXB}$

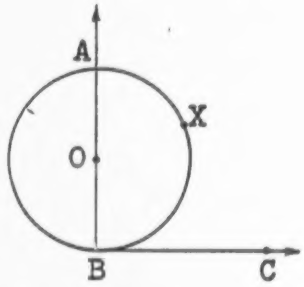


Case I : The secant ray contains a diameter.
Case II : The center of the circle lies in the interior of the angle.
Case III: The center of the circle lies in the exterior of the angle.

Argument for case I

Given: $\overline{AB}$ is a diameter of $\odot O$
$\overrightarrow{BC}$ is tangent to $\odot O$ at B

Prove: $m\angle ABC = \frac{1}{2} m\,\widehat{AXB}$

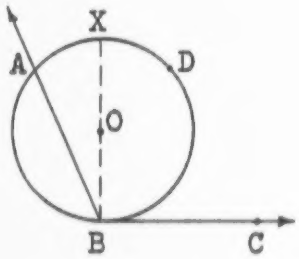


$$\left.\begin{array}{l} \left.\begin{array}{l} \text{①}\ \overline{AB} \text{ is a diameter} \\ \text{②}\ \overrightarrow{BC} \text{ is tangent at } B \end{array}\right\} \xrightarrow{\text{③}} \overline{AB}\perp\overrightarrow{BC} \longrightarrow \ldots \xrightarrow{\text{④}} m\angle ABC = 90 \\ \text{①}\ \overline{AB} \text{ is a diam.} \xrightarrow{\text{⑤}} \widehat{AXB} \text{ is a semicircle} \xrightarrow{\text{⑥}} m\widehat{AXB} = 180 \end{array}\right\} \longrightarrow \ldots \xrightarrow{\text{⑦}}$$

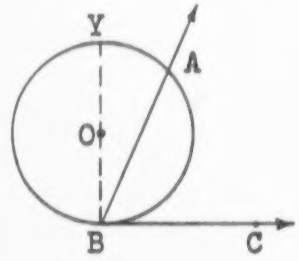
$$\xrightarrow{\text{⑦}} m\angle ABC = \frac{1}{2} m\,\widehat{AXB}$$

$$\therefore \left.\begin{array}{r} \overline{AB} \text{ is a diameter} \\ \overrightarrow{BC} \text{ is tangent at } B \end{array}\right\} \longrightarrow m\angle ABC = \frac{1}{2} m\,\widehat{AXB}$$

Case II: Draw diameter $\overline{BX}$ and use thm 18-11 with case I to add angle measures and arc measures.



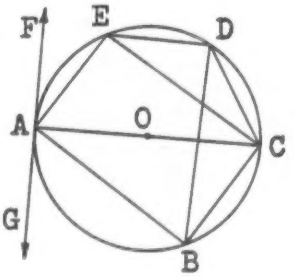
Case III: Draw diameter $\overline{BY}$ and use thm 18-11 with case I to subtract angle measures and arc measures.



EXERCISE SET 18.5

1. Write the reasons for the statements in the proof of case I of theorem 18-11.
2. Write the reasons for the statements in the proof of case II of theorem 18-11.
3. Prove corollary 18-11a.
4. Prove corollary 18-11b.
5. Prove corollary 18-11c.
6. Write the reasons for the statements in the proof of corollary 18-11d.
7. Prove case II of theorem 18-12.

Complete each implication.



8. $\angle DCE$ is an inscribed $\angle \longrightarrow$
9. $\left.\begin{matrix}\angle DEC \text{ intercepts } \widehat{CD}\\ \angle DBC \text{ intercepts } \widehat{CD}\end{matrix}\right\}\longrightarrow$
10. $\widehat{CD} \cong \widehat{AE} \longrightarrow \angle CBD \cong \underline{\ \ ?\ \ }$
11. $\overleftrightarrow{FG}$ tang. to $\odot O$ at $A \longrightarrow m\angle FAE = \underline{\ \ ?\ \ }$
12. $\widehat{ABC}$ is a semicircle $\longrightarrow \angle ABC\ \underline{\ \ ?\ \ }$
13. ABDE is an inscribed quadrilateral $\begin{matrix}\nearrow\\ \searrow\end{matrix}$
14. $\overleftrightarrow{FG}$ tang. to $\odot O$ at $A \longrightarrow m\angle EAG = \underline{\ \ ?\ \ }$
15. $\angle CDE$ is an inscribed angle $\longrightarrow$

16. $\overline{RS}$ is a diameter of $\odot X$, $m\widehat{SV} = 40$, $m\angle QTR = 50$.

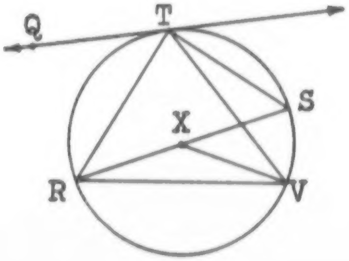
a) $m\angle STV = ?$ f) $m\angle RXV = ?$

b) $m\angle RVX = ?$ g) $m\angle RST = ?$

c) $m\angle RTS = ?$ h) $m\widehat{RV} = ?$

d) $m\widehat{RT} = ?$ i) $m\angle QTV = ?$

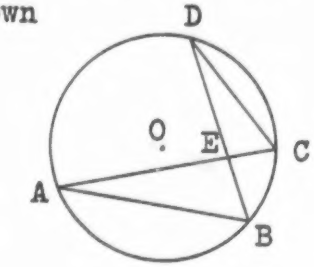
e) $m\angle SRT = ?$ j) $m\widehat{TVR} = ?$



17. The measures of two consecutive angles of an inscribed quadrilateral are 70 and 130. What are the measures of the other two angles?

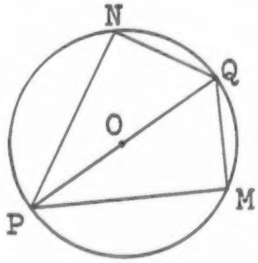
18. Prove: If an inscribed angle is a right angle, then it is inscribed in a semicircle.

19. Given: Chords of $\odot O$ as shown
Prove: $\triangle ABE \sim \triangle DCE$

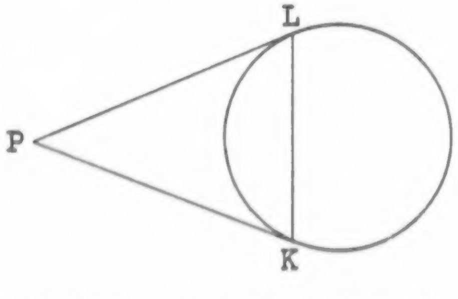


20. Prove: Parallel chords of a circle cut off congruent segments.

21. Given: $\overline{PQ}$ is a diameter of $\odot O$
$\overset{\frown}{QN} \cong \overset{\frown}{QM}$
Prove: $\triangle PQN \cong \triangle PQM$

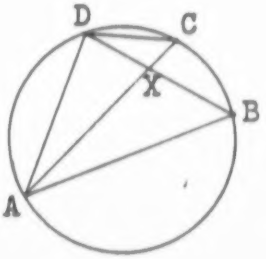


22. Given: $\overline{PK}$ tangent at K
$\overline{PL}$ tangent at L
Prove: $\triangle PKL$ is isosceles

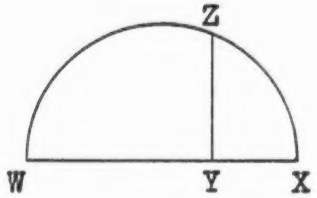


23. Prove: If a parallelogram is inscribed in a circle, it is a rectangle.

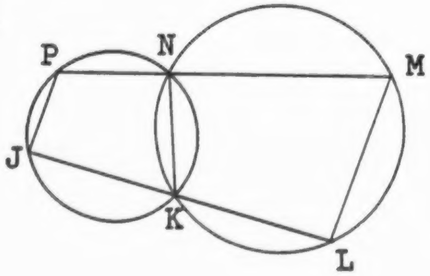
24. Given: $\overset{\frown}{BC} \cong \overset{\frown}{CD}$
Prove: $\triangle ABX \sim \triangle ACD$



25. Given: $\overset{\frown}{WZX}$ is a semicircle
$\overline{ZY} \perp \overline{WX}$
Prove: ZY is the geometric mean between WY and XY

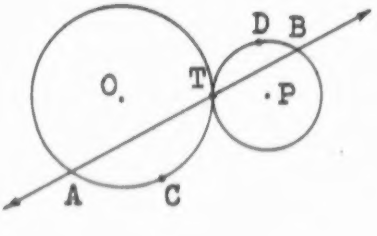


26. Given: $\overline{KN}$ is a common chord
J, K, L are collinear
M, N, P are collinear
Prove: $\overline{LM} \parallel \overline{JP}$

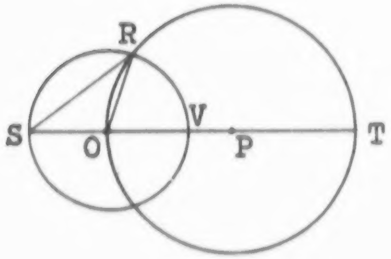


27. Prove: If an inscribed angle intercepts a minor arc, it is acute.

28. Given: ⊙O and ⊙P tangent at T
$\overleftrightarrow{AB}$ contains T
Prove: $m\,\overset{\frown}{ACT} = m\,\overset{\frown}{BDT}$



29. Given: ⊙O and ⊙P
⊙P contains O
Prove: $m\angle RSO = \frac{1}{4} m\,\overset{\frown}{RT}$



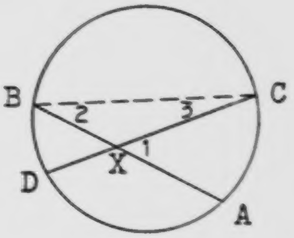
Other Angle and Arc Relations

The next four theorems involve every combination of ways that the measure of an angle can be related to the measures of two intercepted arcs of a circle. We will prove the $\overline{\text{first}}$ one and suggest a plan for the other three. Notice how powerful the use of corollary 13-11d is in these proofs.

Theorem 18-13: The measure of an angle formed by the intersection of two chords in the interior of a circle is half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

Given: $\overline{AB}$ and $\overline{CD}$ are chords intersecting at X

Prove: $m\angle 1 = \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$



Argument

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① Draw $\overline{BC}$

$$\left.\begin{array}{l} ②\ m\angle 2 = \frac{1}{2} m\widehat{AC} \\ ③\ m\angle 3 = \frac{1}{2} m\widehat{BD} \end{array}\right\} \xrightarrow{④} m\angle 2 + m\text{\_}3 = \frac{1}{2} m\widehat{AC} + \frac{1}{2} m\widehat{BD} \xrightarrow{⑤}$$

$$\left.\begin{array}{r} \xrightarrow{⑤} m\angle 2 + m\angle 3 = \frac{1}{2}(m\widehat{AC} + m\widehat{BD}) \\ ⑥\ m\angle 1 = m\angle 2 + m\angle 3 \end{array}\right\} \xrightarrow{⑦} m\angle 1 = \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$$

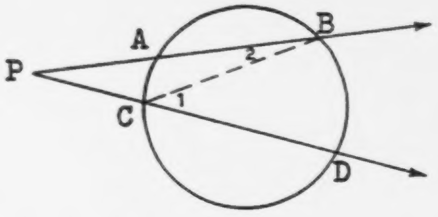
$\therefore$ chords $\overline{AB}$ and $\overline{CD}$ intersect at X $\longrightarrow m\angle 1 = \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$

The proof is similar for any one of the four angles at X.

Theorem 18-14: The measure of an angle formed by two secant rays with vertex in the exterior of the circle is half the difference of the measures of the intercepted arcs.

Given: Secant rays $\overrightarrow{PB}$ and $\overrightarrow{PD}$

Prove: $m\angle P = \frac{1}{2}(m\overset{\frown}{BD} - m\overset{\frown}{AC})$

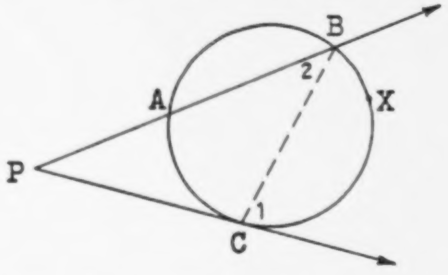


Draw $\overline{BC}$. Use the fact that $m\angle P + m\angle 2 = m\angle 1 \longrightarrow m\angle P = m\angle 1 - m\angle 2$ along with theorem 18-11.

Theorem 18-15: The measure of an angle formed by a secant ray and a tangent ray with vertex in the exterior of the circle is half the difference of the measures of the intercepted arcs.

Given: Secant ray $\overrightarrow{PB}$
$\overrightarrow{PC}$ is tangent at C

Prove: $m\angle P = \frac{1}{2}(m\overset{\frown}{BXC} - m\overset{\frown}{AC})$

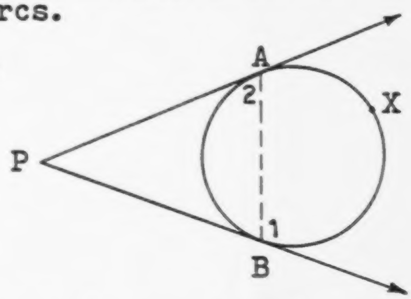


Similar to theorem 18-14, using theorems 18-11 and 18-12.

Theorem 18-16: The measure of an angle formed by two tangent rays to a circle is half the difference of the measures of the intercepted arcs.

Given: $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are tangent at A and B

Prove: $m\ P = \frac{1}{2}(mAXB - mAB)$



Similar to theorem 18-14, using theorem 18-12.

EXERCISE SET 18.6

1. Give the reason for statement ⑥ in the proof of theorem 18-13.
2. Complete the proof of theorem 18-14.
3. Prove theorem 18-15.
4. Prove theorem 18-16.

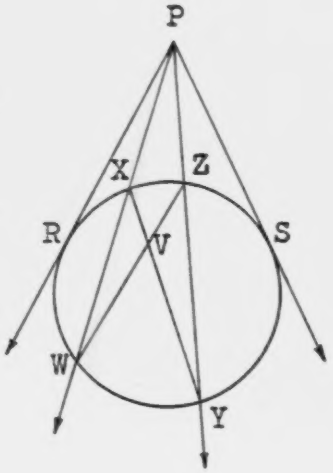
Complete each implication

5. $\overrightarrow{PR}$ tangent at $R \longrightarrow m\angle RPW =$

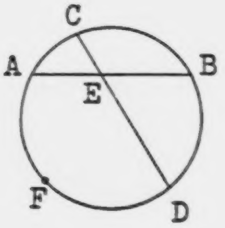
6. $\overline{WZ}$ and $\overline{XY}$ intersect at $V \longrightarrow m\angle YVZ =$

7. $\left.\begin{array}{l}\overrightarrow{PR}\text{ tangent at }R\\ \overrightarrow{PS}\text{ tangent at }S\end{array}\right\}\longrightarrow m\angle RPS =$

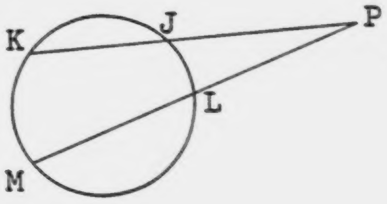
8. $\overrightarrow{PW}$ and $\overrightarrow{PY}$ are secant rays $\longrightarrow m\angle WPY =$



9. If $m\overset{\frown}{AC} = 20$, $m\overset{\frown}{BD} = 70$, find $m\angle AEC$.
10. If $m\angle AED = 130$, $m\overset{\frown}{BC} = 80$, find $m\overset{\frown}{AFD}$.
11. If $m\overset{\frown}{AC} = x$, $m\overset{\frown}{BD} = 2x$, $m\angle AEC = x + 20$, find $m\angle BED$.



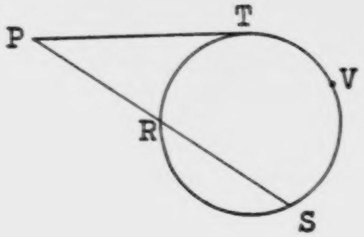
12. If $m\overset{\frown}{KM} = 70$, $m\overset{\frown}{JL} = 20$, find $m\angle P$.
13. If $m\overset{\frown}{JK} = 110$, $m\overset{\frown}{KM} = 60$, $m\overset{\frown}{ML} = 160$, find $m\angle P$.
14. If $m\angle P = 20$, $m\overset{\frown}{KM} = 3x$, $m\overset{\frown}{JL} = x + 10$, find $m\overset{\frown}{KM}$ and $m\overset{\frown}{JL}$.



15. If $m\overset{\frown}{TVS} = 170$, $m\overset{\frown}{TR} = 80$, find $m\angle P$.

16. If $m\angle P = 50$, $m\overset{\frown}{TR} = 75$, find $m\overset{\frown}{RS}$.

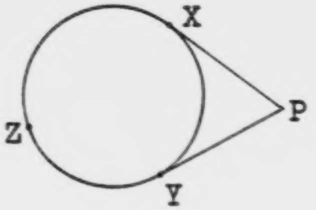
17. If $m\overset{\frown}{TVS} = 210$, $m\angle P = m\overset{\frown}{TR}$, find $m\angle P$.



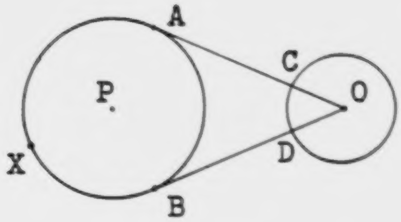
18. If $m\overset{\frown}{XY} = 100$, find $m\angle P$.

19. If $m\angle P = 65$, find $m\overset{\frown}{XZY}$.

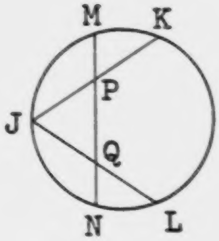
20. If $m\overset{\frown}{XZY} = 4 \cdot m\overset{\frown}{XY}$, find $m\angle P$.



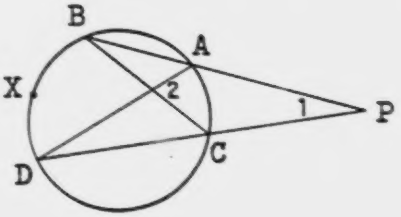
21. Given: $\odot O$ and $\odot P$
$\overline{OA}$ tang. $\odot P$ at A
$\overline{OB}$ tang. $\odot P$ at B
Prove: $m\overset{\frown}{AXB} = m\overset{\frown}{AB} + 2 \cdot m\overset{\frown}{CD}$



22. Given: $\overset{\frown}{JM} \cong \overset{\frown}{JN}$, $\overset{\frown}{MK} \cong \overset{\frown}{NL}$
Prove: $\triangle JQP$ is isosceles



23. Given: P, A, B collinear
P, C, D collinear
Prove: $m\angle 1 + m\angle 2 = m\overset{\frown}{BXD}$



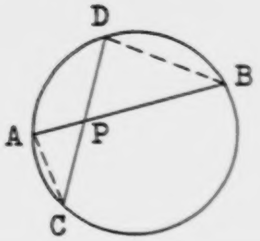
Theorems About Segment Lengths

The figures may look familiar, but the conclusions will be about the relationships between lengths of segments instead of angle and arc measures. Your intuition does not help with such conclusions, so they must be studied carefully. The conclusions of the previous theorems and corollaries of this chapter can now be used to prove similar triangles to aid us in the new proofs.

Theorem 18-17: If two chords intersect in the interior of a circle, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other.

Given: $\overline{AB}$ and $\overline{CD}$ intersect at P

Prove: $PA \cdot PB = PC \cdot PD$



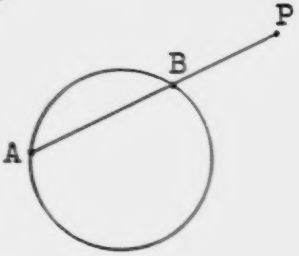
Argument

① Draw $\overline{AC}$ and $\overline{BD}$

$$\left.\begin{array}{l} \text{②}\ \angle A \cong \angle D \\ \text{③}\ \angle C \cong \angle B \end{array}\right\} \xrightarrow{\text{④}} \triangle ACP \sim \triangle DBP \xrightarrow{\text{⑤}} \frac{PA}{PD} = \frac{PC}{PB} \xrightarrow{\text{⑥}} PA \cdot PB = PC \cdot PD$$

$\therefore\ \overline{AB}$ and $\overline{CD}$ intersect at P $\longrightarrow PA \cdot PB = PC \cdot PD$

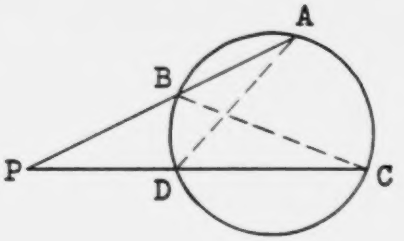
If $\overline{PA}$ is a secant segment of a circle intersecting the circle at A and B, then $\overline{PB}$ is called the external segment of the secant.



Theorem 18-18: If two secant segments have a common endpoint in the exterior of a circle, the product of the length of one segment and the length of its external segment equals the product of the length of the other secant segment and the length of its external segment.

Given: $\overline{PA}$ intersects the circle at A and B
$\overline{PC}$ intersects the circle at C and D

Prove: $PA \cdot PB = PC \cdot PD$



Argument

---

① Draw $\overline{AD}$ and $\overline{BC}$

$$\left.\begin{array}{l} \textcircled{2}\ \angle PAD \cong \angle PCB \\ \textcircled{3}\ \angle APD \cong \angle CPB \end{array}\right\} \xrightarrow{\textcircled{4}} \triangle PAD \sim \triangle PCB \xrightarrow{\textcircled{5}} \frac{PA}{PC} = \frac{PD}{PB} \xrightarrow{\textcircled{6}} PA \cdot PB = PC \cdot PD$$

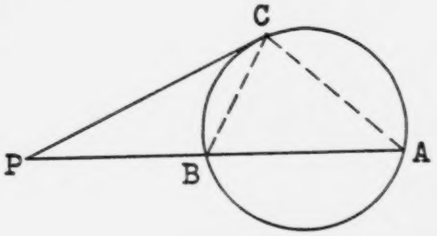
$\therefore$ PA and PB are secant segments $\longrightarrow PA \cdot PB = PC \cdot PD$

Notice that this conclusion is the same form as the conclusion for theorem 18-17. This means that the corresponding products are equal whether P is an interior point or an exterior point of the circle.

Theorem 18-19: If a tangent segment and a secant segment have a common endpoint in the exterior of a circle, the length of the tangent segment is the geometric mean between the length of the secant segment and the length of its external segment.

Given: $\overline{PA}$ intersects the circle at A and B
$\overline{PC}$ is tangent at C

Prove: $\frac{PA}{PC} = \frac{PC}{PB}$



Draw $\overline{CA}$ and $\overline{CB}$ to complete the proof. (Exercise 2)

EXERCISE SET 18.7

1. Write the reasons for statements ② and ③ in the proof of theorem 18-17.

2. Prove theorem 18-19.

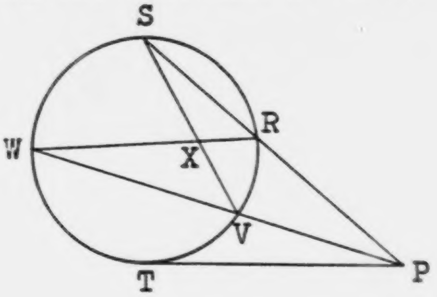
Complete each implication using the theorems of this section.

3. $\overline{SV}$ and $\overline{RW}$ intersect at X →

4. $\overline{PT}$ tangent at T
   $\overline{PS}$ is a secant } →

5. $\overline{PS}$ and $\overline{PW}$ are secants →

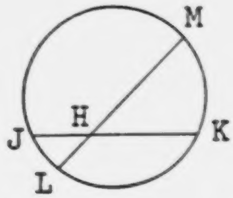
6. $\overline{PW}$ is a secant
   $\overline{PT}$ tangent at T } →



7. If JH = 4, HK = 9, MH = 12, find HL.

8. If JH = x, HK = x + 2, MH = 12, HL = 4, find JH and HK.

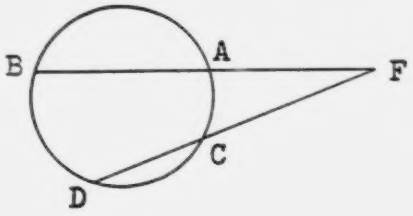
9. If LH = 3, HM = 8, JK = 10, find JH.



10. If FB = 10, FA = 4, FD = 8, find FC.

11. If FA = 8, AB = 8, FC = 10, find CD.

12. If FC = 7, FD = 12, AB = 8, find AF.



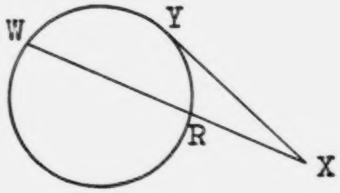
$\overline{XY}$ tangent at Y.

13. If XW = 20, XR = 5, find XY.

14. If XY = 6, XW = 9, find XR.

15. If XR = 5, RW = 7, find XY.

16. If RW = 10, XY = $5\sqrt{3}$, find RX.

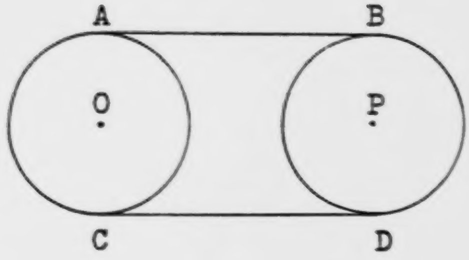


17. Prove: Tangent segments to a circle from an exterior point are congruent. (Draw an auxiliary secant and use theorem 18-19.)

18. If two circles each lie in the exterior of the other, the common internal tangent segments are congruent.

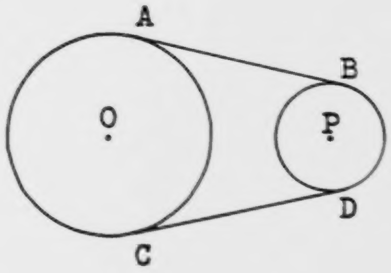
19. Given: $\odot O \cong \odot P$
$\overline{AB}$ is a common tang. at A and B
$\overline{CD}$ is a common tang. at C and D

Prove: $\overline{AB} \cong \overline{CD}$



20. Given: $\odot O \ncong \odot P$
$\overline{AB}$ is a common tang. at A and B
$\overline{CD}$ is a common tang. at C and D

Prove: $\overline{AB} \cong \overline{CD}$



21. In a circle of diameter 18, the distance of a point X from the center is 3. $\overline{MN}$ is any chord of the circle containing X.

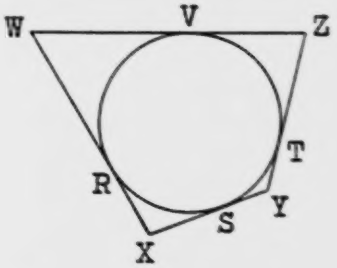
a) Find $MX \cdot NX$.

b) What is the length of the longest chord through X?

c) What is the length of the shortest chord through X?

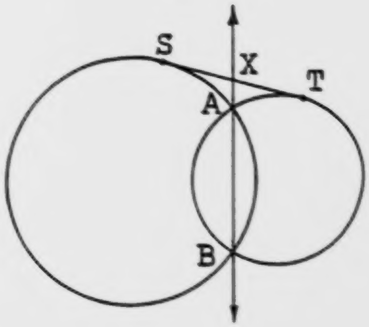
22. Given: WXYZ is a circumscribed quadrilateral

Prove: $WX + YZ = ZW + XY$



23. Given: $\overline{ST}$ is a common tangent of two circles which intersect at A and B

Prove: $\overleftrightarrow{AB}$ bisects $\overline{ST}$



Spheres

The definition of a sphere is the same as that for a circle without the restriction to a plane.

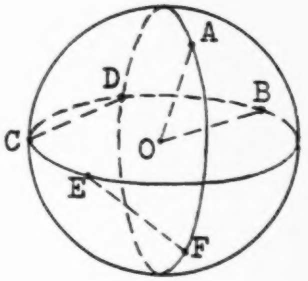
A sphere is the set of all points in space at a given distance from a given point. The given point is the center of the sphere.

We will discuss the sphere briefly in this section to familiarize you with some basic definitions and properties. You may find it interesting and worthwhile to make a more detailed study of the sphere in available references which treat it thoroughly.

As you would suspect, many of the definitions in connection with the sphere are the same as those for a circle with the necessary adaptations to space.

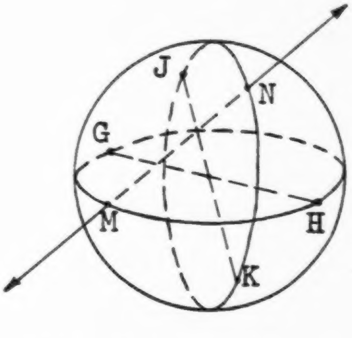
A radius of a sphere is a segment from the center to any point of the sphere. $\overline{OA}$ and $\overline{OB}$ are radii. A is the outer endpoint of $\overline{OA}$.

A chord of a sphere is a segment whose endpoints are on the sphere. $\overline{CD}$ and $\overline{EF}$ are chords.



A diameter of a sphere is a chord that contains the center. $\overline{GH}$ and $\overline{JK}$ are diameters.

A secant of a sphere is a line that contains a chord. $\overleftrightarrow{MN}$ is a secant.



Let O be the center of a sphere with the length of a radius equal to r.

The interior of the sphere is the set of all points X, such that $OX < r$.

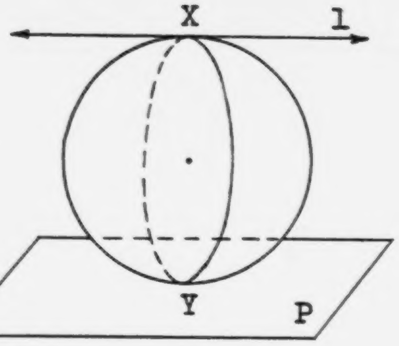
The exterior of the sphere is the set of all points Y, such that $OY > r$.

Two or more spheres are concentric if they have the same center.

A line is tangent to a sphere if it intersects the sphere in exactly one point.
l is tangent at X.

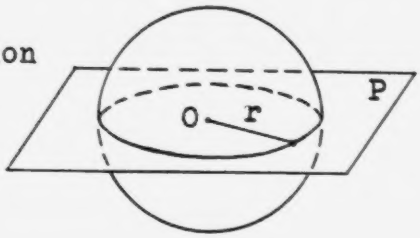
A plane is tangent to a sphere if it intersects the sphere in exactly one point.
P is tangent at Y.

What would be true of every line in P passing through Y?



Theorem 18-20: The intersection of a sphere with a plane through its center is a circle with the same center and the same radius.

If O is the center of the sphere with radius r, and P contains O, then every point in the intersection must be at a distance r from O to satisfy the definition of a sphere. Since every point in the intersection must also lie in plane P, this satisfies the definition of a circle.

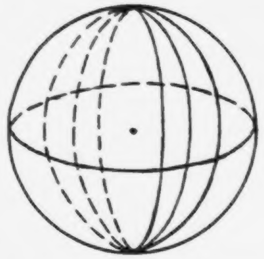


∴ The intersection is the circle with center O and radius r.

A circle determined in this way on a sphere has special significance and a special name.

The intersection of a sphere with a plane through its center is a great circle of the sphere.

Considering the earth as a reasonable model of a sphere, the equator is an example of a great circle, and so are the meridians.

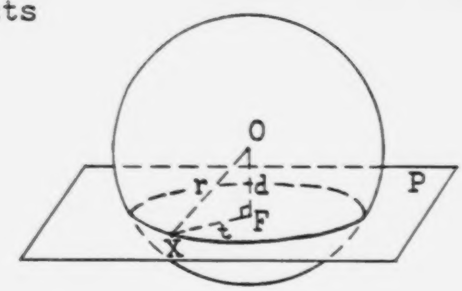


If a plane intersects the interior of a sphere but not the center, the intersection will also be a circle but with a smaller radius. This can be formally proved as a theorem, but we will discuss it as a problem.

Let P be a plane which intersects the interior of a sphere, but does not contain the center O.

Let F be the foot of the perpendicular from O to P.

Let X be any point in the intersection of the plane and the sphere.



Then OX is the radius of the sphere, FX the radius of the circle.

Let $OX = r$, $FX = t$, and $OF = d$. By the Pythagorean Theorem,

$$t^2 + d^2 = r^2 \longrightarrow t^2 = r^2 - d^2 \longrightarrow t = \sqrt{r^2 - d^2}$$

Putting it all together, we have the following.

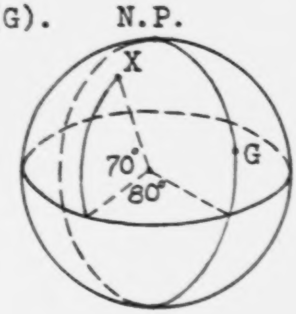
If a plane intersects the interior of a sphere but not the center, the intersection is a circle. The center of the circle is the foot of the perpendicular from the center of the sphere to the plane of the circle. The radius of the circle is $\sqrt{r^2 - d^2}$ where r is the radius of the sphere and d is the distance from the center of the sphere to the center of the circle.

EXERCISE SET 18.8

Use intuition or induction to answer 1-8.

1. Through a given exterior point of a sphere there are how many tangent planes?
2. Through a given point of a sphere there are how many tangent planes?
3. Through a given point of a plane there are how many tangent spheres?
4. Each of three spheres lies in the exterior of the other two. How many planes exist which are common tangents to all three spheres?
5. If two spheres intersect in more than one point, their intersection is _____.
6. From a given exterior point of a sphere any two tangent segments are _____.
7. If a plane is tangent to a sphere, a radius drawn to the point of tangency is _____.
8. The segment joining the intersections of any two great circles of a sphere is a _____.

9. The great circles shown are the equator of the earth and the prime meridian through Greenwich (G). Give the latitude and longitude of point X.



10. Consider the earth as a sphere with a 4000 mile radius. What is the radius of a circle on the surface of the earth all of whose points are at 30° N latitude?

11. The radius of a sphere is 15. The distance from the center to a plane is 12. Find the radius of the intersection of the sphere and the plane.

12. A chord of a sphere has length 16. If the distance from the center of the sphere to the chord is 4, what is the radius of the sphere?

13. The radius of a sphere is 10. Two parallel planes intersect the sphere on opposite sides of the center in circles whose radii are 5 and 8. Find the distance between the planes.

14. The distance between two parallel planes is 18. A sphere is tangent to one of the planes and intersects the other in a circle with radius 12. If the center of the sphere is between the planes, what is the radius of the sphere? (Hint: Use two variables and write two equations.)

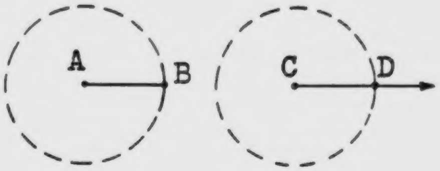
CHAPTER 19 - CONSTRUCTION PROBLEMS AND LOCUS
or FASTEST COMPASS IN THE WEST

Justifying a Construction

With all the theorems and properties in geometry that we now have available, there are a ton of construction problems that we can produce by combining our knowledge of these theorems with the basic constructions of chapter 2.

Our purpose in extending the study of the classic constructions with unmarked straightedge and compass is twofold. The first is reinforcement and enlightenment. Many students have gained a firmer grasp of a particular theorem by performing a construction involving the use of that theorem. Others have "seen the light" on a theorem that was previously obscured.

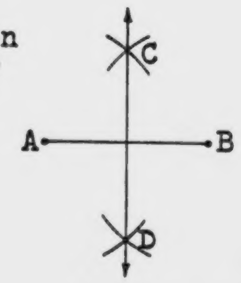
The second purpose relates to the "successful failures" of history. Perhaps Columbus is the most notable example of an explorer who failed to discover what he was looking for, but in the search, made another important discovery. This phenomenon has occurred many times in science and mathematics. In particular, the mathematicians of antiquity made tremendous efforts to perform certain challenging constructions. Sometimes they failed because the constructions were impossible in the first place. But in their search, they discovered and created some branches of mathematics that are very important in today's world. You may very well learn some things in your attempts at construction that you really weren't looking for or expecting.

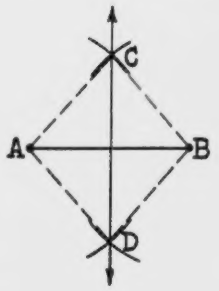


When we strike an arc on a line in doing a construction, remember this is an arc of a circle which we generally do not draw in its entirety. If construction 1 of chapter 2 were shown with the entire circles drawn, the circles would be congruent circles. Then $\overline{CD} \cong \overline{AB}$ because radii of congruent circles are congruent. We use this construction many times to mark off segments congruent to other segments.

When we produce a construction, we must be able to justify that it actually has the relationship or property that we claim.

Consider construction 2 of chapter 2, the construction of the perpendicular bisector of a segment. We can easily justify that $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$ as shown in the figure on the right.





Draw $\overline{AC}$, $\overline{BC}$, $\overline{AD}$, and $\overline{BD}$.

$$\left.\begin{array}{l} \textcircled{1}\ AC = BC \\ \textcircled{2}\ AD = BD \end{array}\right\} \xrightarrow{\textcircled{3}} \overleftrightarrow{CD} \text{ is the } \perp \text{ bisector of } \overline{AB}$$

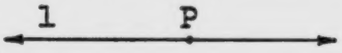
① ② Radii of ≅ circles are ≅.
③ Theorem 11-8: In a plane, if two points are each equidistant from the endpoints of a segment, the line containing them is the perpendicular bisector of the segment.

The justifications for constructions 3 through 6 will be left as exercises.

The constructions that we will consider now may be an end in themselves or they may be tools to use in the construction of more involved problems. As each construction is developed, we will justify it at the same time.

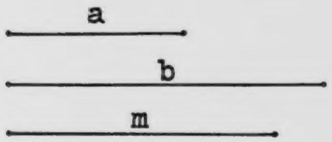
EXERCISE SET 19.1

1. Justify construction 3.
2. Justify construction 4.
3. Justify construction 5.
4. Justify construction 6.
5. The following is another approach to construction 3. Construct a perpendicular to a line at a point on the line.
   (1) With any point O, not on l, as center, construct a circle intersecting l at P and at any second point A.
   (2) Draw $\overline{AO}$ to intersect the circle at a point B.
   (3) Draw $\overline{PB}$.
   (4) Prove $\overline{PB} \perp l$.

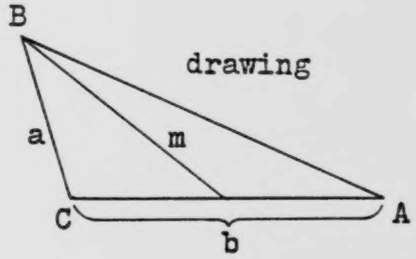


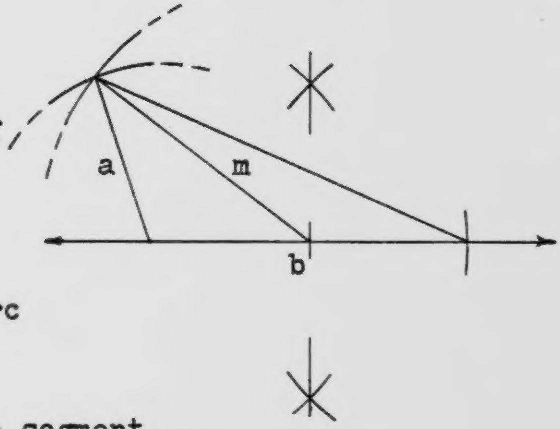
When attempting a construction in which a procedure is not obvious, make a drawing of what the finished figure should look like and study it for relationships that will suggest a procedure.

Example: Construct a triangle given the lengths of two sides, a and b, and the length of the median, m, to the longer side, b.



Vertex B must be at a distance m from the midpoint of $\overline{AC}$ and also at a distance a from point C (or A).





Procedure

(1) Draw a line and mark off a length equal to b.

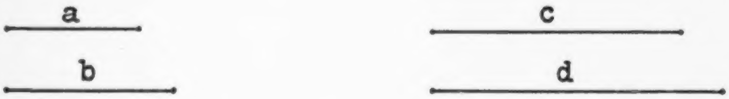
(2) Construct the midpoint of that segment.

(3) With the midpoint as center and m as radius, draw a circle. (or an arc of a circle that will obviously contain the third vertex)

(4) With one endpoint of the segment from step (1) as center and a as radius, strike an arc intersecting the arc of step (3).

(5) From the intersection of the two arcs draw segments to the endpoints of the segment with length b. This completes the desired triangle.

Use constructions 1-6 together with your accumulated knowledge in each of the following. Copy the segments shown and use as required. You do not need to write a justification unless you are challenged to do so.

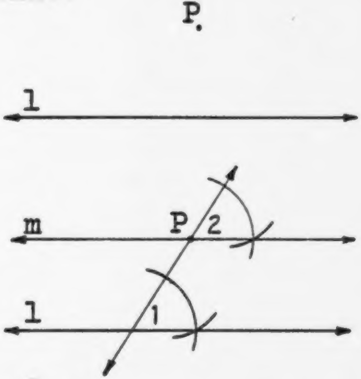


6. Construct an isosceles right triangle with leg of length b.
7. Construct a right triangle with hypotenuse of length d and one acute angle with measure 75.
8. Construct a rhombus with diagonals of lengths a and c.
9. Construct an isosceles triangle with base of length a and altitude to the base of length b.
10. Construct a triangle with two sides of lengths c and d and altitude to the third side of length b.
11. Construct any right triangle. On one leg locate a point by construction which is equidistant from the hypotenuse and from the vertex of the right angle.

Constructions Involving Parallels and Proportions

Construction 7: Construct a parallel to a line through a point not on the line.

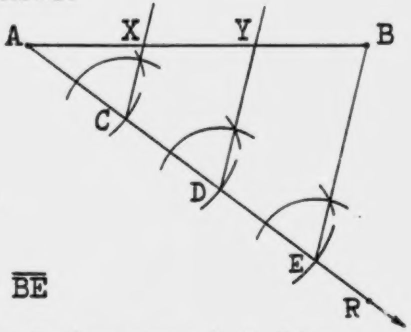
Construct a line through P and parallel to l.



Using our knowledge of parallel lines and transversals, we may introduce a transversal and construct either congruent corresponding angles or congruent alternate interior angles.

Draw any line through P intersecting l.
At P construct an angle congruent to $\angle 1$ using const. 6.
Then $m \parallel l$. Theorem 13-3: If 2 lines are cut by a transversal so that corresp. $\angle$s are $\cong$, the lines are $\parallel$.

Construction 8: Divide a given segment into any number of congruent segments.



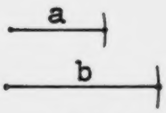
Divide $\overline{AB}$ into 3 congruent segments.

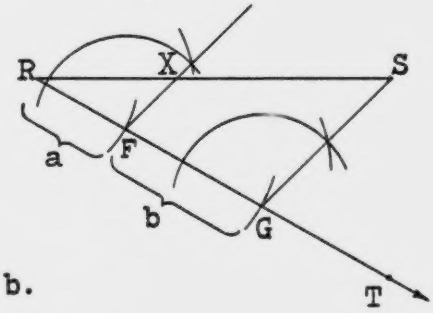
Draw any ray $\overrightarrow{AR}$.
With any convenient radius, mark off three congruent segments on $\overrightarrow{AR}$. $\overline{AC} \cong \overline{CD} \cong \overline{DE}$.
Draw $\overline{BE}$
At D and C construct parallels to $\overline{BE}$ intersecting $\overline{AB}$ at X and Y.
Then $\overline{AX} \cong \overline{XY} \cong \overline{YB}$. Theorem 14-8: If 3 (or more) ∥ lines cut off ≅ segments on one transversal, they cut off ≅ segments on every transversal.

To divide $\overline{AB}$ into n congruent segments, mark off n congruent segments on $\overrightarrow{AR}$ and proceed as above.

Construction 9: Divide a given segment into two segments whose lengths are in a given ratio.

Divide $\overline{RS}$ into two segments such that $\frac{RX}{XS} = \frac{a}{b}$





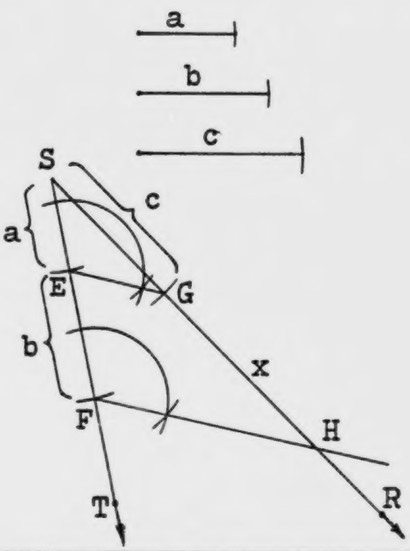
Draw any ray $\overrightarrow{RT}$.
On $\overrightarrow{RT}$ mark off $RF = a$ and $FG = b$.
Draw $\overline{GS}$
At F construct a parallel to $\overline{GS}$ intersecting $\overline{RS}$ at X.
Then $\frac{RX}{XS} = \frac{a}{b}$. Theorem 16-10: If a line is ∥ to one side of a △ and intersects the other 2 sides in distinct points, it divides them proportionally.

Construction 10: Given three segments whose lengths are the first three terms of a proportion, construct a segment whose length is the fourth term of the proportion.

Construct a segment of length x such that $\frac{a}{b} = \frac{c}{x}$

Draw any convenient $\angle RST$.
On $\overrightarrow{ST}$ mark off $SE = a$ and $EF = b$
On $\overrightarrow{SR}$ mark off $SG = c$
Draw $\overline{EG}$
At F construct a parallel to $\overline{EG}$ intersecting $\overrightarrow{SR}$ at H.
Then $GH = x$

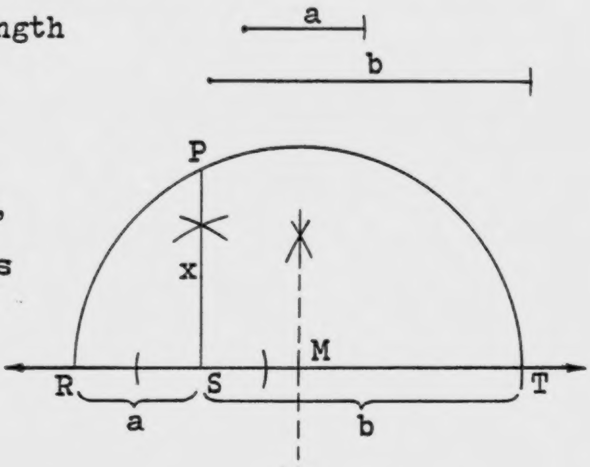
The justification is the same as that for construction 9.



Construction 11: Construct a segment whose length is the geometric mean between the lengths of two given segments.

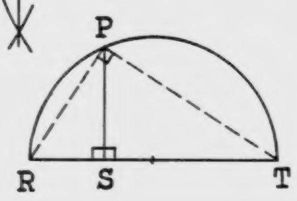
Construct a segment of length x such that $\frac{a}{x} = \frac{x}{b}$

On a line mark off $RS = a$ and $ST = b$.
Construct the midpoint, M, of $\overline{RT}$.
With M as center and MR as radius, construct a semicircle.
At S construct a perpendicular to $\overline{RT}$ intersecting the semicircle at P.
Then $SP = x$.



Justification: Draw $\overline{PR}$ and $\overline{PT}$. $\angle RPT$ is inscribed in a semicircle and thus is a rt. $\angle$. $\triangle RPT$ is a rt. $\triangle$ with $\overline{PS}$ the altitude to the hypotenuse.

By cor. 17-8b, $\frac{RS}{SP} = \frac{SP}{ST}$.

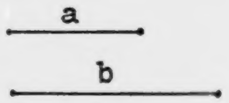


EXERCISE SET 19.2

1. Draw any $\triangle ABC$. Through the midpoint of $\overline{BC}$ construct a line parallel to $\overline{AC}$. State a theorem with your constructed conditions as hypothesis which would relate the parallel to $\overline{AB}$.
2. Draw any $\triangle ABC$. Through each vertex construct a line parallel to the opposite side. What relationship exists between $\triangle ABC$ and the triangle formed by the intersections of the three parallels? What relationship exists between the points A, B, and C and the sides of the new triangle?
3. Draw a segment and divide it into 5 congruent segments.
4. Draw a segment and divide it into two segments whose lengths are in the ratio 1 to 2.

Some constructions can be approached by writing a statement equivalent to the given condition.

Example: Construct a segment of length x such that $x = \sqrt{2ab}$.

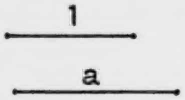


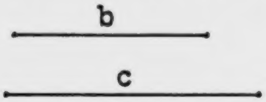
Since all numbers must be positive,

$$x = \sqrt{2ab} \longleftrightarrow x^2 = 2ab \longleftrightarrow \frac{2a}{x} = \frac{x}{b} \quad \text{or} \quad \frac{a}{x} = \frac{x}{2b}$$

Construct the geometric mean between 2a and b or between a and 2b.

Copy the given segments and construct a segment of length x such that x satisfies the given condition.





5. $x = \frac{ab}{c}$
6. $x = \frac{ac}{b}$
7. $x = \sqrt{bc}$
8. $x = ab$
9. $x = 2\sqrt{ab}$
10. $x = \frac{c}{a}$
11. $x = \frac{a^2}{b}$
12. $x = \sqrt{(c+a)(c-a)}$
13. Divide a segment of length c into two segments whose lengths are in the ratio $\frac{a}{b}$.

14. Construct a segment of length $b\sqrt{3}$ in two different ways as suggested by the following.

    (1) $x = b\sqrt{3} \longleftrightarrow x^2 = 3b^2 \longleftrightarrow \frac{b}{x} = \frac{x}{3b}$

    (2) If b is the length of the shorter leg of a 30 - 60 - 90 Δ, $b\sqrt{3}$ is the length of the longer leg.

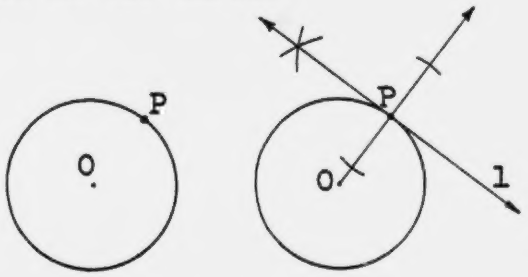
    Compare the lengths of $b\sqrt{3}$ in the two constructions.

Constructions Involving Circles

Construction 12: Construct the tangent to a circle at a given point on the circle.

Construct the tangent to $\odot O$ at P.

Draw $\overline{OP}$
At P construct a $\perp$, l, to $\overline{OP}$.
Then l is tangent to $\odot O$ at P.

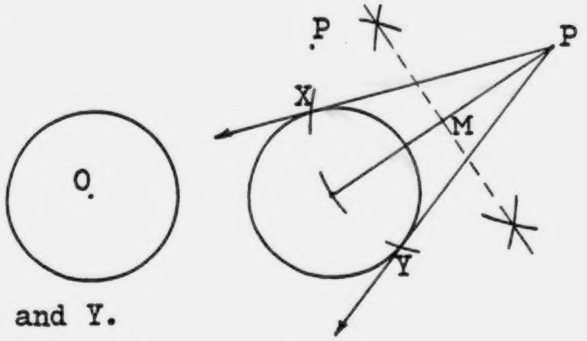


Theorem 18-3: A line in the plane of a circle and $\perp$ to a radius at its outer endpoint is tangent to the circle.

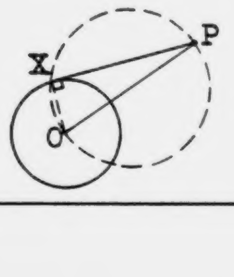
Construction 13: Construct the tangents to a circle through a given exterior point.

Construct the tangents to $\odot O$ through P.

Draw $\overline{OP}$
Construct the midpoint, M, of $\overline{OP}$.
With M as center and MO as radius, strike arcs on the circle at X and Y.
Draw $\overrightarrow{PX}$ and $\overrightarrow{PY}$.
Then $\overrightarrow{PX}$ and $\overrightarrow{PY}$ are tangent to $\odot O$ at X and Y.

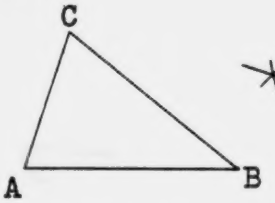


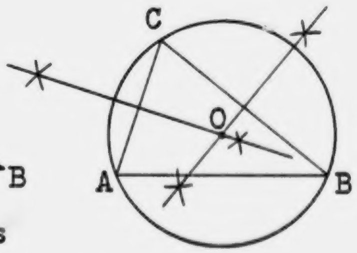
Justification: Draw semicircle $\widehat{OXP}$ and $\overline{OX}$. $\angle OXP$ is inscribed in a semicircle and thus is a rt. $\angle$. Theorem 18-3 again requires $\overline{PX}$ to be tangent.



Construction 14: Circumscribe a circle about a given triangle.

Circumscribe a circle about △ABC.





Construct the perpendicular bisectors of any two sides to intersect at O.
With O as center and OA as radius, draw ⊙O.
Then ⊙O contains A, B, and C and is circumscribed about △ABC.
Theorem 11-7: If a point is on the ⊥ bisector of a segment, it is equidistant from the endpoints of the segment.

Since O is on the ⊥ bisector of $\overline{AC}$ it is equidistant from A and C.

Since O is on the ⊥ bisector of $\overline{BC}$ it is equidistant from B and C.

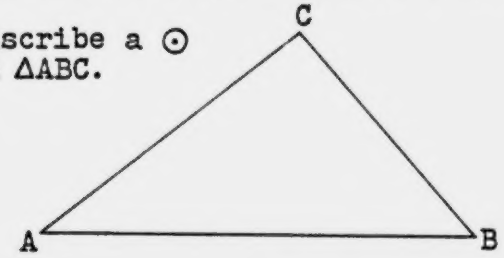
∴ O is equidistant from A, B, and C.

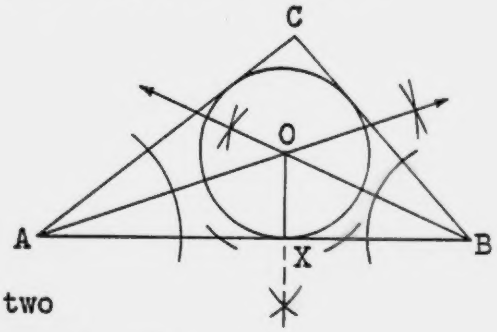
Note: If the ⊥ bisector of $\overline{AB}$ were constructed it would necessarily contain O since the ⊥ bisector of a segment contains all points equidistant from the endpoints of that segment.

The point O is called the *circumcenter* of △ABC.

Construction 15: Inscribe a circle in a given triangle.

Inscribe a ⊙ in △ABC.



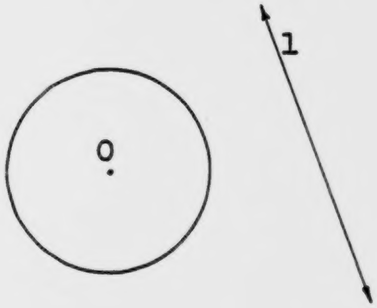


Construct the ∠ bisectors of any two angles to intersect at O.
Construct a ⊥ from O to $\overline{AB}$ to intersect $\overline{AB}$ at X.
With O as center and OX as radius, draw ⊙O.
Then $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are tangent to ⊙O and ⊙O is inscribed in △ABC. The point O is called the *incenter* of △ABC.

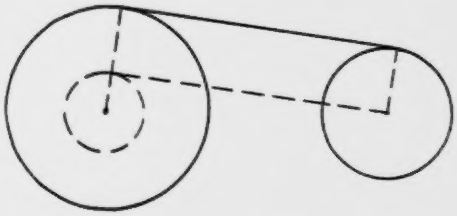
Justification: $\overline{AB}$ is tangent to ⊙O because $\overline{AB}$ ⊥ radius $\overline{OX}$ at X. You complete the justification by showing why $\overline{AC}$ and $\overline{BC}$ must be tangent to ⊙O. (*Hint*: Draw the other perpendiculars from O to $\overline{AC}$ and $\overline{BC}$.)

EXERCISE SET 19.3

1. Draw a circle and construct a tangent to the circle at a point on the circle.
2. Draw a circle and construct two tangents to the circle from an exterior point.
3. Draw an acute triangle and construct its circumscribed circle. Where is the center?
4. Construct a right triangle and construct its circumscribed circle. Where is the center?
5. Draw an obtuse triangle and construct its circumscribed circle. Where is the center?
6. Draw any triangle and construct its inscribed circle.
7. Copy the figure and construct a tangent to $\odot O$ parallel to l.
8. Copy the figure and construct a tangent to $\odot O$ perpendicular to l.



9. Construct a square and circumscribe a circle about it.
10. Construct a rhombus and inscribe a circle in it.
11. Construct an isosceles triangle given the base and the radius of the inscribed circle.
12. Construct an equilateral triangle given the radius of the inscribed circle.
13. Construct a 30 - 60 - 90 triangle given the radius of the inscribed circle.
14. Construct a common external tangent to two nonintersecting noncongruent circles. Study the figure for clues.

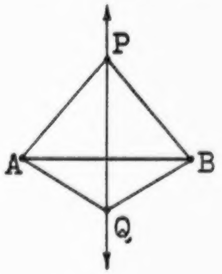


15. Construct a common internal tangent to two nonintersecting noncongruent circles.

Locus

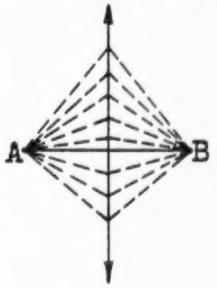
Some of the theorems and constructions we have worked with have been concerned with a condition that a set of points must have in a given situation. Consider the statement of theorem 11-8.

In a plane, if two points are each equidistant from the endpoints of a segment, the line containing them is the perpendicular bisector of the segment.



The condition is that the two points, P and Q, must each be equidistant from A and B. If we place this condition on "all" points in the plane, we could make the following statement.

The set of all points in a plane, each equidistant from the endpoints of a given segment, is the perpendicular bisector of the segment.



Instead of saying that the two points, P and Q, "determine" the line, we are saying that the set of all such points "is" the line. In this sense, the line is called a locus, which is defined as follows.

A locus is the set of all points that satisfy a given condition.

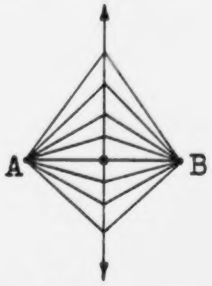
The word locus is Latin for place. The plural of locus is loci (lo-sigh).

For a figure to be a locus for a given condition, the following requirements must be met.

1. Every point of the locus must satisfy the condition.
2. Every point that satisfies the condition must be on the locus.

For the above example, (1) every point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment, and (2) every point in the plane that is equidistant from the endpoints of a segment is on the perpendicular bisector of the segment.

The next example is very similar. What is the locus of points in a plane that are vertices of the vertex angle of all isosceles triangles in the plane with a given segment as base?



If $\overline{AB}$ is the given segment, the locus would again appear to be the perpendicular bisector of $\overline{AB}$. Requirement (2) is met since, for a point to be a vertex of the vertex angle of an isosceles triangle with $\overline{AB}$ as base, it must be on the perpendicular bisector of $\overline{AB}$. Is requirement (1) met by the perpendicular bisector? How about the midpoint of $\overline{AB}$? Since the vertices of a triangle cannot be collinear, the midpoint of $\overline{AB}$ is not in the locus. The answer to the original question would be as follows.

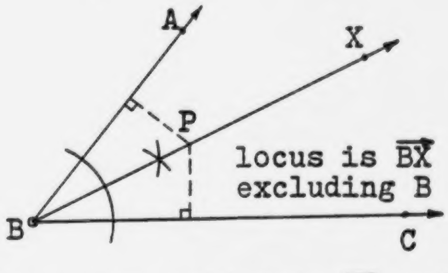
The locus of points in a plane that are vertices of the vertex angle of all isosceles triangles in the plane with a given segment as base is the perpendicular bisector of the segment, excluding its midpoint.

A locus may or may not be restricted to a plane. It is important to give this careful consideration in any locus problem.

The answer to a locus problem may be given as a precise word description of the locus as a figure, or by a drawing with the locus clearly identified. Sometimes it is appropriate to construct the locus. If the locus is immediately obvious, it may be drawn or constructed directly.

Example: What is the locus of points in the interior of an angle equidistant from the sides of the angle?

The locus is the ray that bisects the angle, excluding its endpoint.

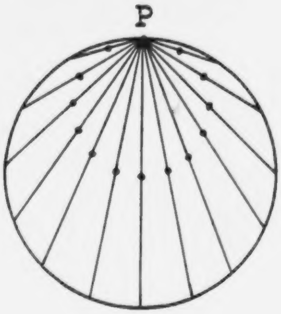


Every point, P, of the locus is equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Every point in the interior of $\angle ABC$ equidistant from $\overrightarrow{BA}$ and $\overrightarrow{BC}$ is on the locus.

If the locus is not immediately obvious, we experiment inductively. Locate individual points that satisfy the condition until a pattern emerges (hopefully). Then complete the drawing and description of the locus.

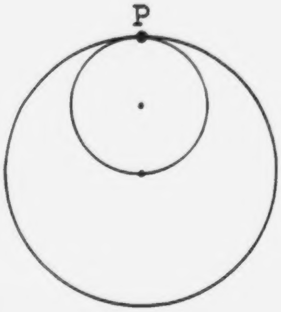
Example: What is the locus of the midpoints of all chords of a circle with a given point of the circle as their common endpoint?

Draw a number of chords with a common endpoint, P, and by construction or measuring, locate their midpoints. Any bright lights flashing???



From this we should be able to give a complete drawing and description of the locus.

The locus is a circle, excluding point P. Its center is the midpoint of the segment from P to the center of the given circle. (Why?) Its radius is half the radius of the given circle. (Why?)



Since all this is speculation based on our induction, we would have to prove that (1) every point of the circle except P is the midpoint of a chord with P as one endpoint, and (2) the midpoint of every chord of the given circle with P as one endpoint lies on the locus circle. We won't prove it here, but it can be done.

EXERCISE SET 19.4

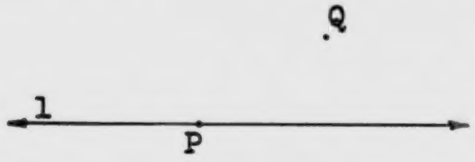
Make a drawing and give a word description for each of the following.

1. What is the locus of points in a plane equidistant from two parallel lines in the plane?
2. What is the locus of points equidistant from two parallel planes?
3. What is the locus of points in a plane 2 inches from a given line in the plane?
4. What is the locus of points 2 inches from a given line?
5. What is the locus of the centers of all circles in a plane tangent to a given line at a given point of the line?
6. What is the locus of the midpoints of all radii of a given circle?
7. What is the locus of points in a plane equidistant from two given points in the plane?
8. What is the locus of points in a plane equidistant from two intersecting lines in the plane?
9. What is the locus of the midpoints of all chords of a given length less than the diameter in a given circle?
10. What is the locus of points in a plane 1 inch from a given point in the plane?
11. What is the locus of points in the exterior of a circle which are endpoints of all tangent segments of a given length?
12. What is the locus of points in space equidistant from three noncollinear points?
13. What is the locus of the midpoints of all chords of a circle containing a given interior point of the circle not the center?
14. What is the locus of midpoints of all segments joining one vertex of a triangle to points of the opposite side?
15. What is the locus of the vertices of right angles of all right triangles in a plane with a given segment in the plane as hypotenuse?
16. What is the locus of points 4 inches from a given plane and at the same time 5 inches from a given point in the plane?

Locus Used in Construction

Sometimes a construction problem can be approached by finding an intersection of loci.

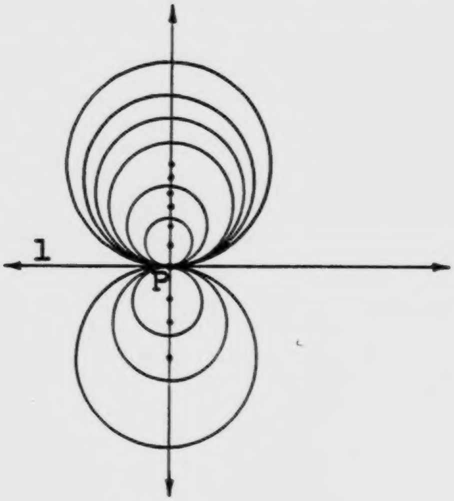
Example: Given a line l, a point P on l, and a point Q not on l, construct a circle tangent to l at P and passing through Q.



All we need to do is find the center of the circle. Consider the two conditions of the problem independently from one another.

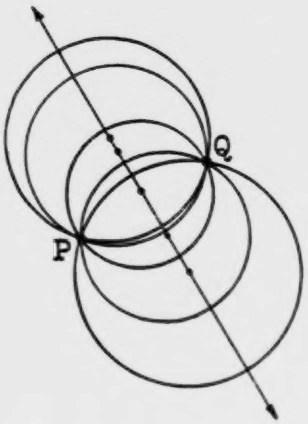
1. What is the locus of the centers of all circles in a plane and tangent to l at P?

The locus is the line in the plane perpendicular to l at P, excluding P.

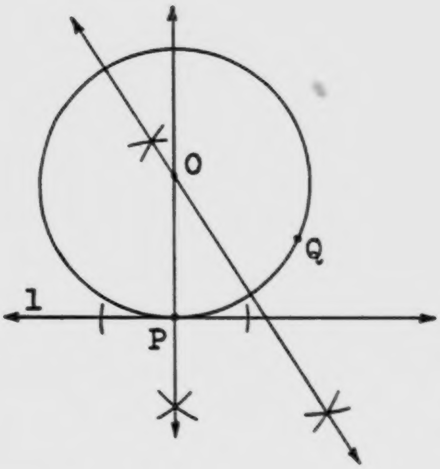


2. What is the locus of the centers of all circles in a plane passing through two points, P and Q?

The locus is the perpendicular bisector of $\overline{PQ}$ in the plane.



Since the center of the desired circle must satisfy both conditions, it must be the intersection of the two loci. We construct the two loci which intersect at O and draw the circle.



EXERCISE SET 19.5

1. Draw a segment and an angle. Construct a circle whose radius is the length of the given segment and that is tangent to both sides of the given angle.
2. Draw two parallel lines and indicate any point between them. Construct a circle tangent to both lines and containing the given point.
3. Draw a segment, a line, and a circle. Construct a circle whose radius is the length of the given segment and that is tangent to both the given line and the given circle.
4. Draw three segments of unequal lengths. Construct a parallelogram such that a diagonal is congruent to the longest segment and the sides are congruent to the other two segments.
5. Draw three segments of unequal lengths. Construct a triangle such that a side is congruent to the longest segment, the altitude to that side is congruent to the shortest segment, and the median to that side is congruent to the third segment.
6. Draw a segment and an acute angle. Construct an isosceles triangle such that the base is congruent to the given segment and the vertex angle is congruent to the given angle.

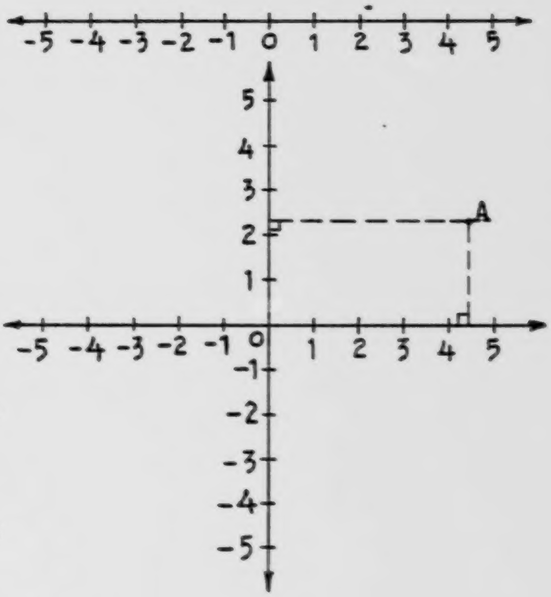
CHAPTER 20 - PLANE COORDINATE GEOMETRY
or LET'S GET COORDINATED

The Coordinate Plane

The study of geometry is concerned with both the content of the subject and the method or approach to the subject. Throughout its history, the content has grown as new discoveries are made, but the method was essentially the Euclidean approach from the period of the Greeks till about 1637. That was the year in which the Frenchman, Rene Descartes, published a paper which had the effect of performing a marraige of geometry and algebra. He and fellow Frenchman, Pierre de Fermat, introduced the world to analytic geometry which associates the points of geometric figures with the numbers of algebra. It is a part of this broad subject that we shall examine briefly in this chapter.

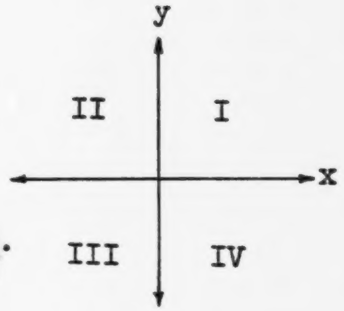
You are probably familiar with the coordinate system from your previous studies, but we will develop it from the beginning to reinforce whatever knowledge you now have.

We have already established a coordinate system on a line in chapter 6, in which there is a one to one correspondence between the points of the line and the real numbers. We will now extend this concept into the plane by drawing another number line perpendicular to the first so that 0 on the second line coincides with 0 on the first. We will require that the "unit" on the first line be the distance used for the unit on the second line with positive numbers above 0 and negative numbers below.



Now we will associate a given point in the plane with a point on each line by dropping the perpendiculars from the given point to each line as shown for point A. Then A is associated with the foot of each perpendicular, or with two numbers, one on each number line.

It is conventional to call the first number x and let it be the coordinate on the horizontal line, which we call the x-axis. The second number is designated y and is the coordinate on the vertical line, or the y-axis. The plural of axis is axes (ak'sēz). The intersection of the two axes is the origin. As you can see, these two axes divide the plane into four nonintersecting regions called quadrants. They are numbered as shown in the figure.

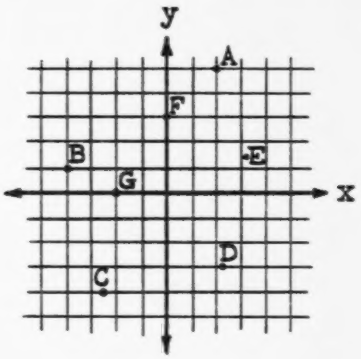


We have established a way to associate every point in the plane with two real numbers in a particular order. We call the two numbers an ordered pair, and we write it (x,y). The plane is now called the coordinate plane, or the Cartesian plane (after Descartes), such that for every point of the plane there is an associated ordered pair of real numbers, and for every ordered pair of real numbers there is an associated point in the plane.

We call the ordered pair the coordinates of the point, and the point is the graph of the ordered pair. We may at times refer to an ordered pair as a point or a point as an ordered pair for convenience, since no confusion can result. It is convenient to use lined graph paper for this purpose in which the lines are associated with the integers.

Some examples of points with coordinates are as follows.

A(2,5)
B(-4,1)
C($-\frac{5}{2}$,-4)
D($\sqrt{5}$,-3)
E(π,$\sqrt{2}$)
F(0,3)
G(-2,0)

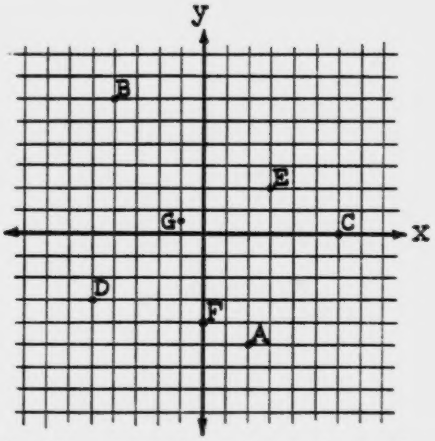


The x-coordinate of a point is called the abscissa.
The y-coordinate of a point is called the ordinate.

The two basic types of problems you are concerned with at this stage are
1. Given a point, read its coordinates.
2. Given an ordered pair, plot its graph.

EXERCISE SET 20.1

1. Write the coordinate of each point.



2. On lined graph paper, plot the graph of each ordered pair. For those that are not integers, approximate as closely as possible.

$A(-4,0)$

$B(-1,-5)$

$C(6,2)$

$D(2\frac{1}{2},-\frac{1}{2})$

$E(0,7)$

$F(-7,2.8)$

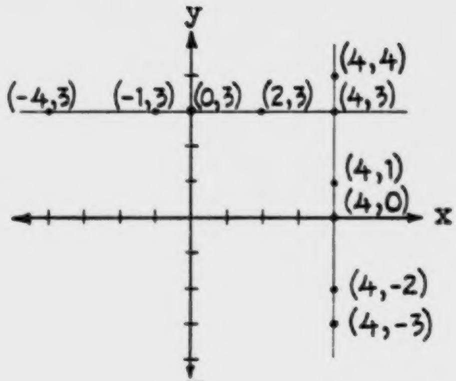
$G(\pi,-6)$

$H(\sqrt{3},\frac{14}{3})$

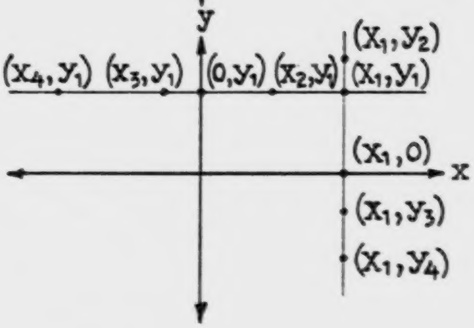
The Use of Subscripts With Coordinates

When we say that the coordinates of a point P are (x,y), this is the representative form of the coordinates of the infinite set of points in the plane. As we investigate the geometric properties relating these points, we will refer to specific fixed points with subscripts, $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, etc. The subscripts allow us to distinguish between points and still retain the conventional notation. Later in the chapter, we will denote fixed points such as (a,b).

Notice the characteristics of the coordinates of all points on the same horizontal line or the same vertical line. On any horizontal line, all points have the same ordinate. On any vertical line, all points have the same abscissa.


y
(4,4)
(-4,3)
(-1,3)
(0,3)
(2,3)
(4,3)
(4,1)
(4,0)
x
(4,-2)
(4,-3)

Therefore, when using subscripts, we may use the same subscript for the abscissa or the ordinate of several points when they are related in this manner as shown here.


y
(x₁,y₂)
(x₄,y₁)
(x₃,y₁)
(0,y₁)
(x₂,y₁)
(x₁,y₁)
(x₁,0)
x
(x₁,y₃)
(x₁,y₄)

The distance between any two points on the same horizontal line is established just as it was in chapter 6 for any two points on the number line.

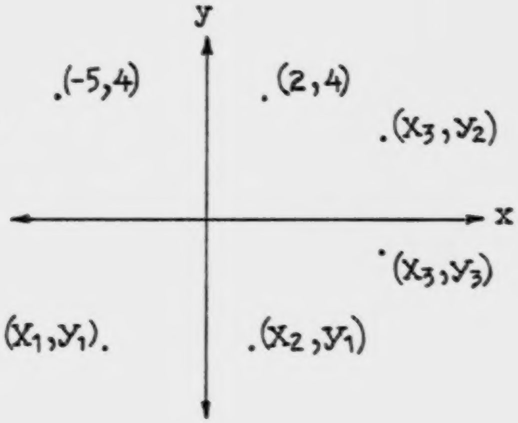
The distance between $(-5,4)$ and $(2,4)$ is the positive difference of the abscissas, $2-(-5)=7$.

Therefore, the distance between the two points whose coordinates are (x_1,y_1) and (x_2,y_1) is

$x_2 - x_1$ if $x_2 > x_1$.

Likewise, the distance between the two points whose coordinates are (x_3,y_3) and (x_3,y_2) is

$y_2 - y_3$ if $y_2 > y_3$.


y
(-5,4)
(2,4)
(x₃,y₂)
x
(x₃,y₃)
(x₁,y₁)
(x₂,y₁)

EXERCISE SET 20.2

1. For each set of points, state whether they would all lie on the same horizontal line, the same vertical line, or neither.

   a) (5,3), (5,-5), (5,0)

   b) (2,2), (-4,-4), (7,7)

   c) (0,-8), (-3,-8), (8,-8)

   d) (x_1,y_1), (x_2,y_1), (x_3,y_1)

   e) (x_3,y_2), (x_3,y_4), (x_3,y_1)

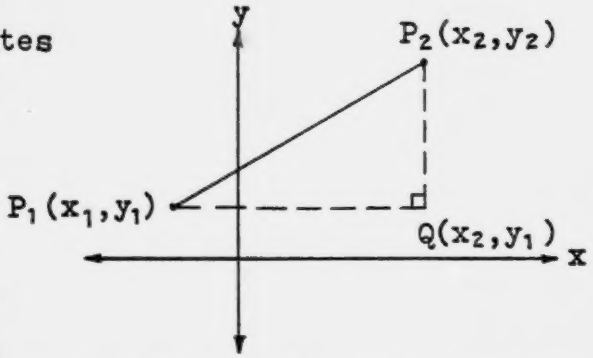
   f) (x_1,y_1), (x_2,y_2), (x_3,y_3)

Write the distance between each pair of points.

2. (6,4) and (6,1)
3. (3,7) and (0,7)
4. (1,-5) and (8,-5)
5. (-2,4) and (-2,-3)
6. (5,-1) and (-2,-1)
7. (-3,-6) and (-3,-2)
8. (x_1,y_2) and (x_1,y_3), $y_3 > y_2$
9. (x_2,y_2) and (x_3,y_2), $x_2 > x_3$

The Distance Formula

It is very important that we be able to express the distance between any two points in the coordinate plane in terms of the coordinates of the points.



Let P_1 and P_2 be any two points with coordinates (x_1,y_1) and (x_2,y_2). If they are on the same horizontal or vertical line, the distance between them was expressed in the previous section. If they are not on the same horizontal or vertical line, we can always draw segments through the points parallel to the axes as shown to form a right triangle. Then the coordinates of Q are (x_2,y_1) for this figure. From the last section, we have

$$P_1Q = x_2 - x_1 \text{ if } x_2 > x_1 \text{ and } P_2Q = y_2 - y_1 \text{ if } y_2 > y_1$$

By the Pythagorean theorem we have

$$(P_1P_2)^2 = (P_1Q)^2 + (P_2Q)^2$$

$$(P_1P_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that $x_2 - x_1$ and $x_1 - x_2$ are always negatives of one another so that when we square both of them, the positive results are the same.

$$(x_2 - x_1)^2 = (x_1 - x_2)^2 \text{ and } (y_2 - y_1)^2 = (y_1 - y_2)^2$$

Therefore, either point may be taken as (x_1,y_1) and the differences of the abscissas and ordinates may be taken in either order.

We write the result as a theorem stating the distance formula.

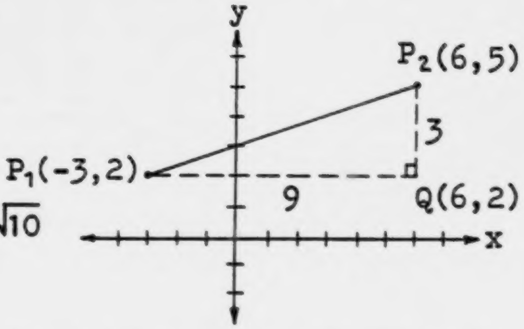
Theorem 20-1: The distance between two points P_1 and P_2 whose coordinates are (x_1,y_1) and (x_2,y_2) is expressed by the formula

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Since this formula is so closely related to the Pythagorean theorem, it helps to emphasize just how powerful that ancient theorem is in mathematics.

When the coordinates are all integers, the legs of the right triangle can easily be found by drawing the figure and counting. From the figure, we have

$P_1P_2 = \sqrt{9^2 + 3^2} = \sqrt{81 + 9} = \sqrt{90} = 3\sqrt{10}$



Of course, the formula allows us to find the distance between any two points without drawing a right triangle and even without plotting the points.

The distance between (5,-3) and (-2,1) is

$d = \sqrt{(-2-5)^2 + (1-(-3))^2} = \sqrt{49 + 16} = \sqrt{65}$

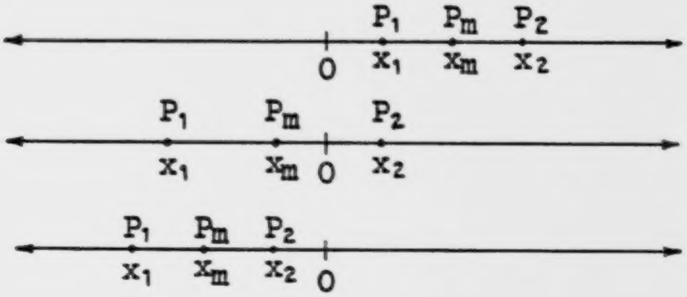
EXERCISE SET 20.3

Find the distance between the given points. If the distance is an irrational number, express in simplest radical form.

1. (2,4) and (6,7)
2. (0,0) and (-5,5)
3. (5,1) and (1,3)
4. (4,-2) and (-3,1)
5. (3,4) and (-4,-2)
6. (-6,3) and (-2,-4)
7. (25,-18) and (19,-15)
8. (a,b) and (2a,2b)
9. (0,0) and (a + b,c)
10. (2a,0) and $(a, a\sqrt{3})$
11. $\triangle ABC$ has vertices A(-3,4), B(-1,-3), and C(4,2). Show that $\triangle ABC$ is isosceles. Which segment is the base?
12. The four consecutive vertices of a quadrilateral are at points (3,0), (5,-2), (6,5), and (2,3). Show that the diagonals are congruent.
13. $\triangle RST$ has vertices R(4,-1), S(7,5), and T(-4,3). Using the converse of the Pythagorean theorem, show that $\triangle RST$ is a right triangle. Which point is the vertex of the right angle?

The Midpoint Formula

As you have observed in the past, midpoints arise frequently in geometry problems. Let's first consider the midpoint of two points on a number line and then we will extend this to the plane.



Let x_1 and x_2 be the coordinates of P_1 and P_2 and let x_m be the coordinate of the midpoint, P_m, of $\overline{P_1P_2}$. If $x_1 < x_2$, the following argument will hold for all cases.

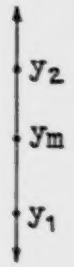
$$\left.\begin{array}{l} P_1P_m = x_m - x_1 \\ P_mP_2 = x_2 - x_m \\ P_1P_m = P_mP_2 \end{array}\right\} \longrightarrow x_m - x_1 = x_2 - x_m \longrightarrow 2\cdot x_m = x_1 + x_2 \longrightarrow$$

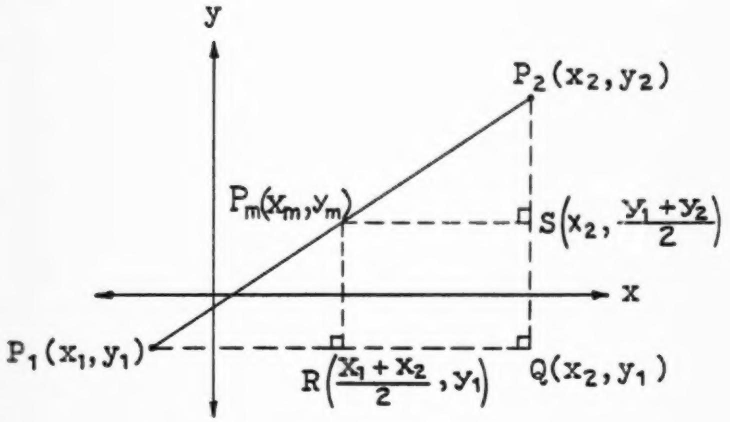
$$\longrightarrow x_m = \frac{x_1 + x_2}{2}$$

If $x_2 < x_1$, we would interchange x_1 and x_2 and still get the same result. In words, we could say that the coordinate of the midpoint is the average of the coordinates of the endpoints.

If the number line were vertical, similar reasoning would lead to the conclusion that

$$y_m = \frac{y_1 + y_2}{2}$$




y
$P_2(x_2,y_2)$
$P_m(x_m,y_m)$
$S\left(x_2, \frac{y_1+y_2}{2}\right)$
x
$P_1(x_1,y_1)$
$R\left(\frac{x_1+x_2}{2}, y_1\right)$
$Q(x_2,y_1)$

Now let P_1 and P_2 be any two points in the coordinate plane, not on the same horizontal or vertical line, with coordinates (x_1,y_1) and (x_2,y_2). Let P_m be the midpoint of $\overline{P_1P_2}$ with coordinates (x_m,y_m). Draw segments through the endpoints and parallel to the axes intersecting at Q as we did before. Now draw a segment through P_m and parallel to the y-axis intersecting $\overline{P_1Q}$ at R, and a segment through P_m and parallel to the x-axis intersecting $\overline{P_2Q}$ at S.

Since P_m is the midpoint of $\overline{P_1P_2}$ and $\overline{P_mR} \parallel \overline{P_2Q}$ and $\overline{P_mS} \parallel \overline{P_1Q}$, R is the midpoint of $\overline{P_1Q}$ and S is the midpoint of $\overline{P_2Q}$.

Then the abscissa of R is $\frac{x_1+x_2}{2}$

and the ordinate of S is $\frac{y_1+y_2}{2}$

But the abscissa of R is the abscissa of P_m and the ordinate of S is the ordinate of P_m. We write this result as a theorem called the midpoint formula.

Theorem 20-2: If P_1 and P_2 are two points with coordinates (x_1,y_1) and (x_2,y_2) and P_m is the midpoint of P_1P_2 with coordinates (x_m,y_m), then

$$(x_m,y_m) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

In problem situations, you may wish to consider the midpoint formula as two separate formulas.

$$x_m = \frac{x_1 + x_2}{2} \quad \text{and} \quad y_m = \frac{y_1 + y_2}{2}$$

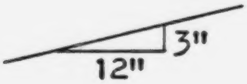
EXERCISE SET 20.4

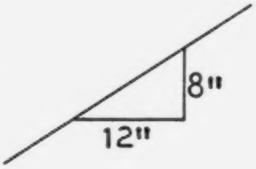
Find the coordinates of the midpoint of the segment joining each pair of points.

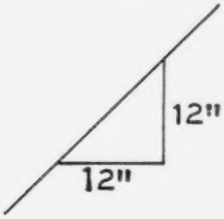
1. (2,1) and (4,5)
2. (8,0) and (0,-6)
3. (-4,3) and (2,5)
4. (3,6) and (-4,-3)
5. (5,-2) and (-3,7)
6. $(2\frac{1}{2},6)$ and $(5\frac{1}{2}, \frac{1}{2})$
7. (-3.2,4) and (5.6,-6.4)
8. (2a,2b) and (4a,4b)
9. (0,0) and (2a + 2b,2c)
10. (2a,0) and (2b,2c)
11. The midpoint of a segment has coordinates (-2,1). If one endpoint has coordinates (0,5), find the coordinates of the other endpoint.
12. The midpoint of a segment has coordinates (3a,b). If one endpoint has coordinates (2a,2b), find the coordinates of the other endpoint.
13. The vertices of a quadrilateral are at points (-3,-2), (4,3), (3,-1), and (-1,3). Find the coordinates of the midpoint of the longer diagonal.
14. △ABC has vertices at points A(7,3), B(-1,6), and C(-3,-1). Find the length of the median from B to $\overline{AC}$.

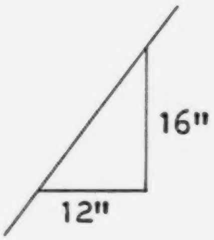
Slope of a Line

A roofer would express the steepness of a roof as its "pitch" by referring to the amount of vertical change, or rise, in one foot of horizontal change, or run. Obviously, as the figures below illustrate, the pitch increases as the rise increases and the run remains constant.







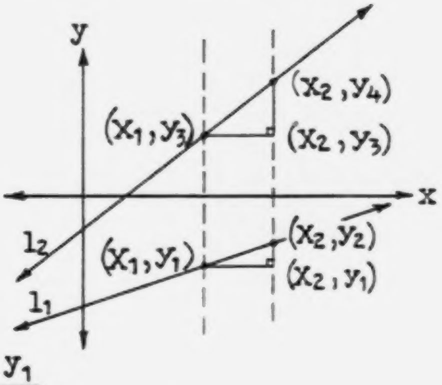


In the geometric sense, we will refer to the steepness of a line as its "slope". We would like our definition to be consistent with the natural idea of the slope of a roof, a roadway, or a ramp. We will define slope to be a ratio of the two changes, rise and run, but we want to be sure to define it so that the ratio, as a number, increases as the steepness increases.

From the appearance of the figure shown, we would say that l_2 is steeper than l_1, so we would like its slope, as a number, to be larger. For the points shown, the run is the same, $x_2 - x_1$.

The rise for l_2 is $y_4 - y_3$.

The rise for l_1 is $y_2 - y_1$.



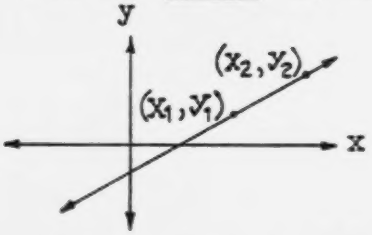
If $y_4 - y_3 > y_2 - y_1$, then $\frac{y_4 - y_3}{x_2 - x_1} > \frac{y_2 - y_1}{x_2 - x_1}$.

That is, the ratio of the rise to the run is larger for l_2 than for l_1 which is what we want it to be. You would find the ratio of the run to the rise to be in conflict with our wishes.

It is conventional to use the symbol m for the numerical value of the slope.

If (x_1, y_1) and (x_2, y_2) are coordinates of two points on a line with $x_1 \neq x_2$, then the slope, m, of the line is defined as

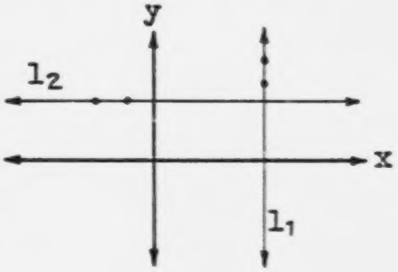
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$



Naturally, if $x_2 = x_1$, then $x_2 - x_1 = 0$ and a zero denominator is meaningless. Since $x_2 = x_1$ for any two distinct points on a vertical line, we must conclude that there is no number that can be the slope of a vertical line by this definition.

For any two distinct points on a horizontal line, $y_1 = y_2$ and

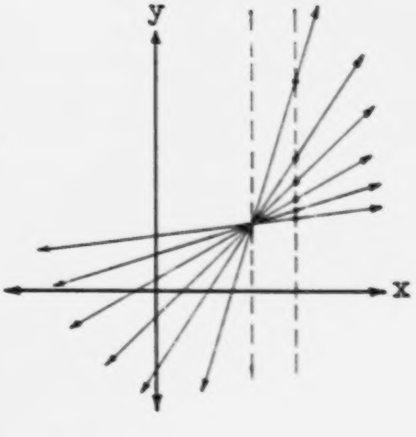
$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0$$



This is an acceptable numerical value for slope. Do not confuse these two statements.

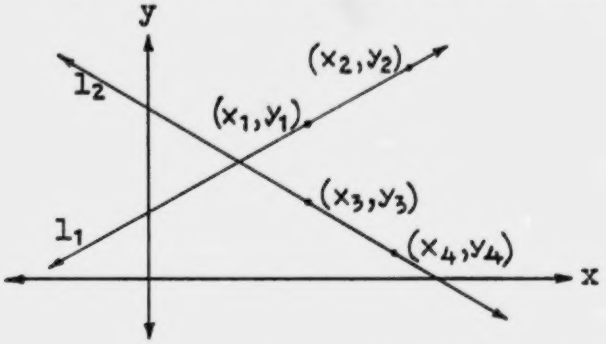
l_1 has no slope.

l_2 has slope 0.



To help keep them straight, consider the same run for several lines as shown here. As the line approaches the horizontal, the rise (numerator) approaches zero and the ratio approaches zero. As the line approaches the vertical, the rise simply increases without bound and doesn't approach any number. So the slope of a vertical line would have to be a number larger than any other number and there is no such number.

Of course, you're impatiently asking, "But what if the line runs the other direction?". In the figure below, l_1 and l_2 seem to have the same steepness, don't they? But if we want the slope to be related to the "direction the line runs", they certainly don't run the same direction.



Fortunately, the definition we've already given takes care of this for us. The definition requires the difference of the ordinates and the difference of the abscissas to be in the same order.

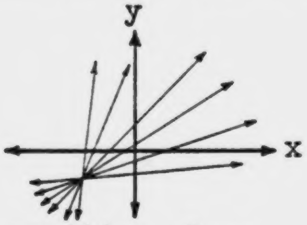
In the figure, $x_2 > x_1$, $y_2 > y_1$, $x_4 > x_3$, $y_3 > y_4$.

If the slopes of l_1 and l_2 are m_1 and m_2 respectively,

then $m_1 = \frac{y_2 - y_1}{x_2 - x_1}$ is positive because numerator and denominator are both positive.

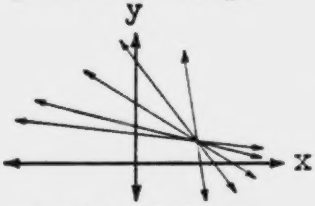
and $m_2 = \frac{y_4 - y_3}{x_4 - x_3}$ is negative because the numerator is negative and the denominator is positive.

Therefore, l_1 has positive slope and all lines with the same coordinate relations as l_1 have positive slope. The line rises from left to right.



positive slope

l_2 has negative slope and all lines with the same coordinate relations as l_2 have negative slope. The line falls from left to right.



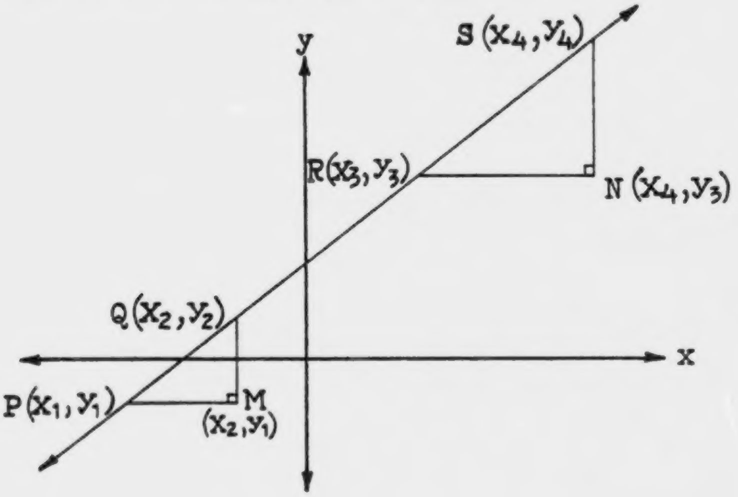
negative slope

Does it really matter which point is taken as (x_1, y_1)? If the names were reversed on any example we've discussed, it would simply change the sign of both numerator and denominator and the sign of the ratio would be the same.

NOTE: The order of the differences must be the same for the slope formula but not necessarily for the distance formula because of the squaring operation. Do not confuse the two.

$$(x_2 - x_1)^2 = (x_1 - x_2)^2 \quad \text{but} \quad x_2 - x_1 \neq x_1 - x_2$$

Any pair of points on a line may be selected to determine its slope. For the figure below, the two triangles can easily be proved similar.



$$\Delta PQM \sim \Delta RSN \longrightarrow \frac{QM}{PM} = \frac{SN}{RN}$$

$$\therefore\ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$$

We will consider the slope of a segment to be the slope of the line containing the segment.

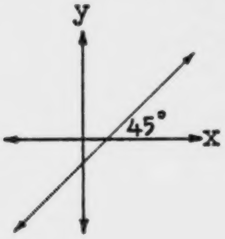
EXERCISE SET 20.5

If the line determined by the given points has a slope, write its slope in simplest form.

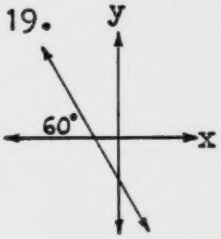
1. $(1,3)$, $(4,5)$
2. $(4,2)$, $(1,6)$
3. $(-3,-2)$, $(5,-2)$
4. $(2,-3)$, $(5,3)$
5. $(0,0)$, $(8,-2)$
6. $(3,2)$, $(-6,5)$
7. $(-4,1)$, $(6,-3)$
8. $(0,4)$, $(-3,0)$
9. $(4,-3)$, $(4,7)$
10. $(-6,3)$, $(6,4)$
11. $(-1,-8)$, $(2,10)$
12. (x_1,y_1), (x_1,y_2)
13. (a,b), $(2a,2b)$
14. $(c-d,d-c)$, (c,d)
15. (x_3,y_3), (x_4,y_3)
16. If a line contains a point in quadrant I and a point in quadrant III, then its slope is (positive, negative).
17. If a line contains a point in quadrant II and a point in quadrant IV, then its slope is (positive, negative).

Using your knowledge of the 45 - 45 - 90 Δ and the 30 - 60 - 90 Δ, write the slope of each line.

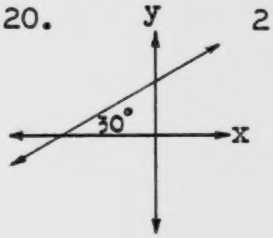
18.



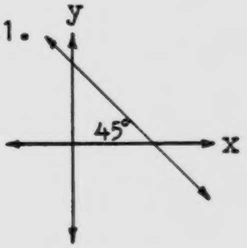
19.



20.



21.



22. Quadrilateral ABCD has vertices at points A(-2,2), B(3,4), C(0,5), and D(-5,3). Find the slopes of all four sides. What kind of quadrilateral is ABCD?
23. ΔABC has vertices at points A(-6,-1), B(2,-5), and C(4,6). Find the slope of the median from C to $\overline{AB}$.
24. $(-2,3)$ and $(3,m)$ lie on a line with slope $\frac{2}{5}$. Find m.
25. $(1,6)$ and $(3,n)$ lie on a line with slope -5. Find n.
26. The slope of a line is -3. If the line contains the point $(0,2)$, plot three other points on the line.
27. The slope of a line is $\frac{3}{4}$. If the line contains the point $(-2,-1)$, plot three other points on the line.

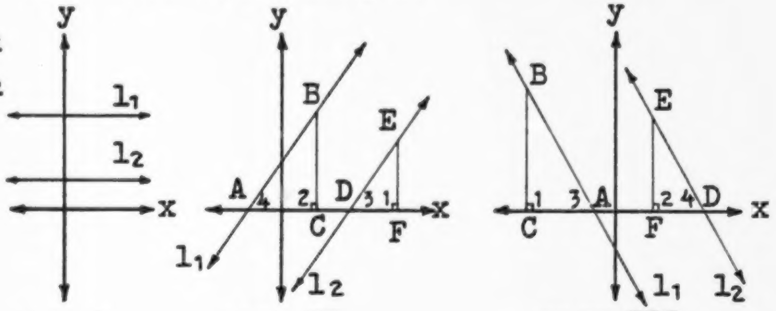
Parallel and Perpendicular Lines

Our intuition strongly suggests that parallel lines have equal slopes and that lines with equal slopes are parallel. These are two important theorems that can be easily proved.

Theorem 20-3: If two nonvertical lines are parallel, they have equal slopes.

Given: $l_1 \parallel l_2$

Prove: $m_1 = m_2$



case I　　case II　　case III

Argument

Case I

If l_1 and l_2 are $\parallel$ to the x-axis, they each have slope 0, so $m_1 = m_2 = 0$.

① If l_1 and l_2 are not $\parallel$ to the x-axis, they both intersect the x-axis. From points B on l_1 and E on l_2, draw $\perp$s to the x-axis. Label all points as shown.

$$\left.\begin{array}{l} \textcircled{1}\ \overline{BC} \perp \text{x-axis} \\ \textcircled{1}\ \overline{EF} \perp \text{x-axis} \end{array}\right\} \to \ldots \xrightarrow{\textcircled{2}} \left.\begin{array}{l} \angle 1 \cong \angle 2 \\ \\ \end{array}\right.$$

$$\left.\begin{array}{r} \ldots \xrightarrow{\textcircled{2}} \angle 1 \cong \angle 2 \\ \textcircled{3}\ l_1 \parallel l_2 \xrightarrow{\textcircled{4}} \angle 3 \cong \angle 4 \end{array}\right\} \xrightarrow{\textcircled{5}} \triangle ABC \sim \triangle DEF \xrightarrow{\textcircled{6}} \frac{BC}{AC} = \frac{EF}{DF}$$

For case II, $m_1 = \frac{BC}{AC}$ and $m_2 = \frac{EF}{DF}$. $\therefore m_1 = m_2$

For case III, $m_1 = -\frac{BC}{AC}$ and $m_2 = -\frac{EF}{DF}$.

$$\textcircled{6}\ \frac{BC}{AC} = \frac{EF}{DF} \xrightarrow{\textcircled{7}} -\frac{BC}{AC} = -\frac{EF}{DF}. \quad \therefore m_1 = m_2$$

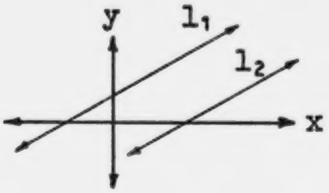
$\therefore$ For nonvertical lines, $l_1 \parallel l_2 \longrightarrow m_1 = m_2$

Before proving the converse, remember that a conditional statement is equivalent to its contrapositive.
$(p \rightarrow q) \longleftrightarrow (\sim q \rightarrow \sim p)$

Theorem 20-4: If two lines have equal slopes, they are parallel.

Given: $m_1 = m_2$

Prove: $l_1 \parallel l_2$

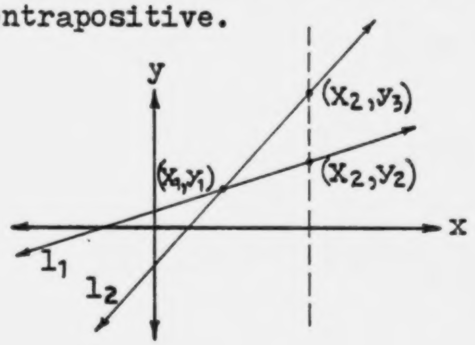


The stated theorem is equivalent to its contrapositive. If two lines are not parallel, they do not have equal slopes.

We will prove the contrapositive.

Given: $l_1 \nparallel l_2$

Prove: $m_1 \neq m_2$



Argument

$l_1 \nparallel l_2 \longrightarrow l_1$ intersects l_2

Draw any vertical line intersecting l_1 and l_2 in distinct points. Label coordinates as shown.

$m_1 = \dfrac{y_3 - y_1}{x_2 - x_1}$ and $m_2 = \dfrac{y_2 - y_1}{x_2 - x_1}$

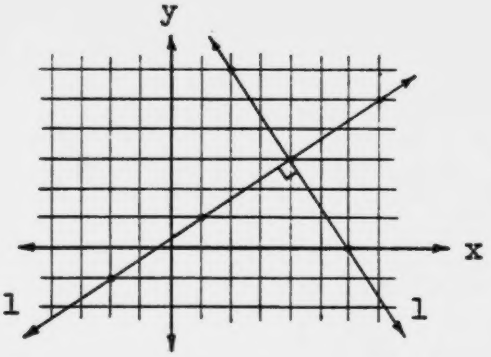
Since $y_3 - y_1 \neq y_2 - y_1$, $m_1 \neq m_2$ because the numerators are $\neq$ and the denominators are $=$

$\therefore l_1 \nparallel l_2 \longrightarrow m_1 \neq m_2$

$(l_1 \nparallel l_2 \longrightarrow m_1 \neq m_2) \longleftrightarrow (m_1 = m_2 \longrightarrow l_1 \parallel l_2)$

$\therefore m_1 = m_2 \longrightarrow l_1 \parallel l_2$

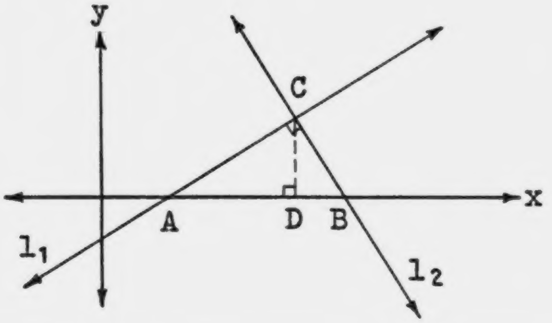
The relationship between the slopes of perpendicular lines is not as obvious. If we examine the numerical example shown, we notice that the slopes of l_1 and l_2 are $\frac{2}{3}$ and $-\frac{3}{2}$ respectively. These numbers are negative reciprocals of one another.



Theorem 20-5: If two nonvertical lines are perpendicular, the slope of each line is the negative reciprocal of the slope of the other.

Given: $l_1 \perp l_2$

Prove: $m_1 = -\frac{1}{m_2}$



Argument

Label the intersection points as shown.

① Draw a $\perp$ from C to intersect the x-axis at D.

② $l_1 \perp l_2 \rightarrow \ldots \xrightarrow{③} \triangle ABC$ is a rt. $\triangle$

① $CD \perp AB$

$\xrightarrow{④} \triangle ADC \sim \triangle CDB \xrightarrow{⑤} \frac{CD}{DA} = \frac{BD}{DC}$

⑥ $m_1 = \frac{CD}{DA}$ $\xrightarrow{⑩}$

⑦ $m_2 = -\frac{DC}{BD}$

⑧ $\frac{BD}{DC} = -\frac{1}{-\frac{DC}{BD}}$

$\xrightarrow{⑨} \frac{BD}{DC} = -\frac{1}{m_2}$

$\xrightarrow{⑩} m_1 = -\frac{1}{m_2}$

$\therefore$ For nonvertical lines, $l_1 \perp l_2 \longrightarrow m_1 = -\frac{1}{m_2}$

Theorem 20-6: If the slopes of two lines are negative reciprocals of each other, the lines are perpendicular.

This converse may be proved by using the figure from theorem 20-5 with $\overline{CD}\perp\overline{AB}$, $m_1 = -\frac{1}{m_2}$, and reversing the steps of that theorem.

EXERCISE SET 20.6

1. The slope of line l is $\frac{5}{4}$. The slope of any line parallel to l is _____, and the slope of any line perpendicular to l is _____.
2. The slope of line m is -3. The slope of any line parallel to m is _____, and the slope of any line perpendicular to m is _____.
3. Given points A(5,6), B(-1,-4), C(h,-2), and D(-1,3). If $\overline{AB} \parallel \overline{CD}$, find h.

Complete each implication using the theorems of this section.

4. $\left.\begin{array}{r}\text{slope of } \overleftrightarrow{EF} = \frac{a}{b} \\ \text{slope of } \overleftrightarrow{GH} = \frac{a}{b}\end{array}\right\} \longrightarrow$

5. $\left.\begin{array}{r}\text{slope of } \overleftrightarrow{AB} = \frac{r}{s} \\ \overleftrightarrow{AB} \perp \overleftrightarrow{CD}\end{array}\right\} \longrightarrow$

6. $\left.\begin{array}{r}\text{slope of } \overleftrightarrow{LM} = \frac{c}{b} \\ \overleftrightarrow{JK} \parallel \overleftrightarrow{LM}\end{array}\right\} \longrightarrow$

7. $\left.\begin{array}{r}\text{slope of } \overleftrightarrow{BC} = \frac{e}{f} \\ \text{slope of } \overleftrightarrow{CD} = -\frac{f}{e}\end{array}\right\} \longrightarrow$

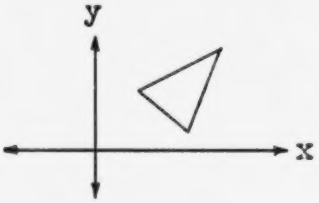
8. $\triangle ABC$ has vertices at points A(3,-2), B(5,4), and C(-4,3). Find the slopes of the three sides and the three altitudes.
9. Given the points R(-6,-2), S(-2,4), and T(c,-2), find the value of c such that ∠RST is a right angle.

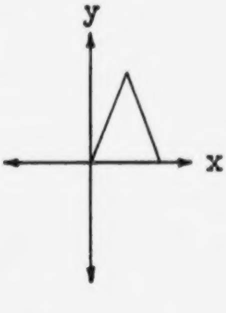
10. Given the points A(-4,1), B(3,0), C(8,5), and D(1,6). Prove that ABCD is a rhombus by showing that the diagonals are perpendicular bisectors of each other. (Suggestion: Show that the diagonals have the same midpoint and that their slopes are negative reciprocals of each other.)

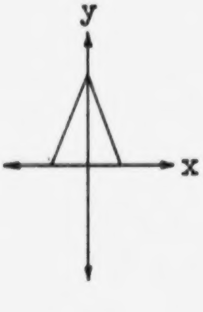
Proving Theorems in Coordinate Geometry

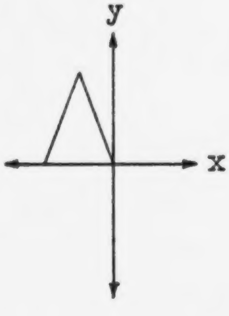
Now that we can express distance, midpoints, and the parallel and perpendicular relations all in terms of coordinates, we are well armed to attack proofs by coordinate methods. A coordinate geometry proof is essentially an algebra proof in which the statements in the argument are in terms of the coordinates. For this reason, we will not make general use of flow proof, but will use it for certain implications where appropriate.

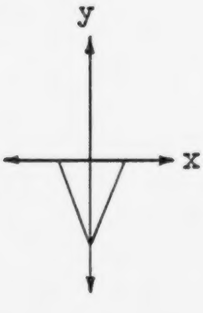
If we should wish to prove a theorem about a given isosceles triangle in the coordinate plane shown here, we would undoubtedly find the algebra very involved. Fortunately, there would always be an isometry transformation under which we could map the triangle onto any of the following triangles in the same plane.





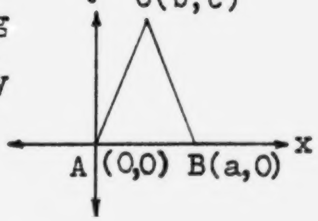






Since the triangles are all congruent, the properties we would be interested in would be invariant. We may as well save ourselves the trouble of making the transformation and simply place the figure in a convenient position with respect to the coordinate axes in the first place. In this sense, we are actually starting with the figure and assigning the coordinate axes to the figure.

Once we have selected a placement, we also have some liberties in assigning coordinates to certain points. We will dispense with the subscripts to simplify the notation.



If $\triangle ABC$ is isosceles with $\overline{AC} \cong \overline{BC}$ and placed as shown, the coordinates of A are (0,0). Let the coordinates of B and C be as shown above.

$$AC = \sqrt{(b-0)^2 + (c-0)^2} = \sqrt{b^2 + c^2}$$

$$BC = \sqrt{(b-a)^2 + (c-0)^2} = \sqrt{(b-a)^2 + c^2}$$

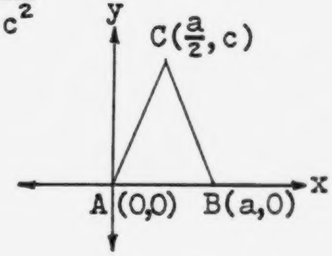
$$\therefore \sqrt{b^2 + c^2} = \sqrt{(b-a)^2 + c^2}$$

$$b^2 + c^2 = (b-a)^2 + c^2$$

$$b^2 + c^2 = b^2 - 2ab + a^2 + c^2$$

$$2ab = a^2$$

$$b = \frac{a}{2}$$



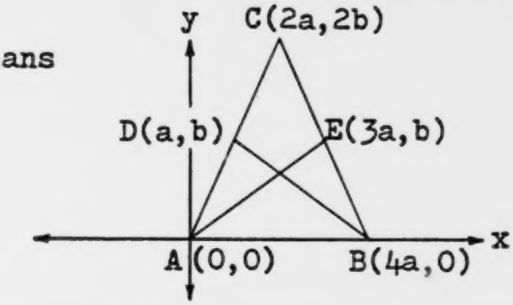
So the abscissa of point C is half the abscissa of point B. If we wish, we may assign coordinates (2a,0) to B and (a,b) to C to avoid fractions. This will be allowed hereafter without proving it each time.

The following example theorems have already been proved by Euclidean methods earlier in the course, so we will not number them here.

Example theorem: If a triangle is isosceles, the medians to the congruent sides are congruent.

Given: $\overline{AC} \cong \overline{BC}$
$\overline{AE}$ and $\overline{BD}$ are medians

Prove: $\overline{AE} \cong \overline{BD}$



Argument

Assign the placement and coordinates as shown.
It is convenient to assign (4a,0) to B in order to avoid fractions throughout the proof.
D and E are assigned directly by the midpoint formula.

$AE = \sqrt{(3a-0)^2 + (b-0)^2} = \sqrt{9a^2 + b^2}$

$BD = \sqrt{(4a-a)^2 + (0-b)^2} = \sqrt{(3a)^2 + b^2} = \sqrt{9a^2 + b^2}$

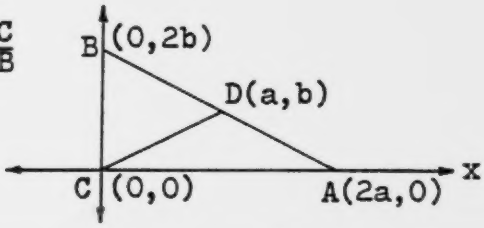
$\therefore\ \overline{AE} \cong \overline{BD}$

NOTE: You do know that $\sqrt{9a^2 + b^2}$ is in simplest form, don't you???

Example theorem: The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Given: $\angle ACB$ is a rt. $\angle$ of ΔABC
D is the midpoint of $\overline{AB}$

Prove: $AD = BD = CD$



Argument

Assign the placement and coordinates as shown.
We are justified in assigning $\overline{CA}$ and $\overline{CB}$ to the x and y axes since it is given that $\angle ACB$ is a rt. $\angle$.

$AD = BD$ by definition of a midpoint.

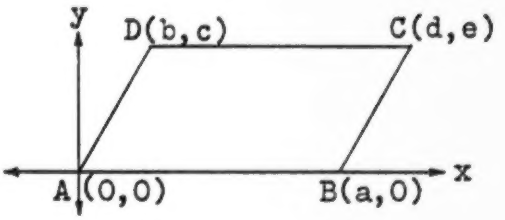
$AD = \sqrt{(2a-a)^2 + (0-b)^2} = \sqrt{a^2 + b^2}$

$CD = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$

$\therefore\ AD = BD = CD$

The relationships of the coordinates of the special quadrilaterals can be established before proving theorems about them. We want to express the coordinates in terms of the fewest number of variables possible. We will establish them for the parallelogram and leave the others as exercises.

Let the placement and coordinates be initially assigned as shown for ▱ABCD. We will denote slopes in the following manner.
Slope of $\overline{AD}$ as $m(\overline{AD})$.



$$\left.\begin{aligned} \overline{DC} \parallel \overline{AB} \longrightarrow m(\overline{DC}) &= m(\overline{AB}) \\ m(\overline{DC}) &= \frac{e-c}{d-b} \\ m(\overline{AB}) &= 0 \end{aligned}\right\} \longrightarrow \frac{e-c}{d-b} = 0 \longrightarrow e-c=0 \longrightarrow e=c$$

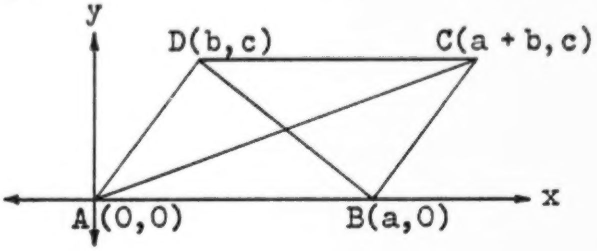
$$\left.\begin{aligned} \overline{BC} \parallel \overline{AD} \longrightarrow m(\overline{BC}) &= m(\overline{AD}) \\ m(\overline{BC}) &= \frac{c-0}{d-a} \\ m(\overline{AD}) &= \frac{c-0}{b-0} \end{aligned}\right\} \longrightarrow \frac{c}{d-a} = \frac{c}{b} \longrightarrow d-a=b \longrightarrow d=a+b$$

In any theorems about parallelograms we may assign the coordinates of point C in terms of the coordinates of B and D.

Example theorem: The diagonals of a parallelogram bisect each other.

Given: ABCD is a ▱

Prove: $\overline{AC}$ and $\overline{BD}$ bisect each other



Argument

Assign the placement and coordinates as shown.

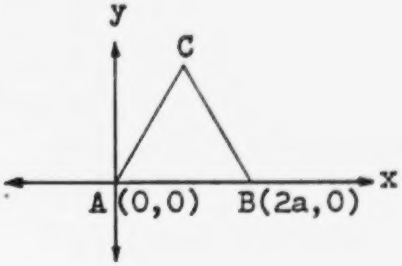
Midpoint of $\overline{BD}$ has coordinates $\left(\frac{a+b}{2}, \frac{0+c}{2}\right) = \left(\frac{a+b}{2}, \frac{c}{2}\right)$

Midpoint of $\overline{AC}$ has coordinates $\left(\frac{(a+b)+0}{2}, \frac{c+0}{2}\right) = \left(\frac{a+b}{2}, \frac{c}{2}\right)$

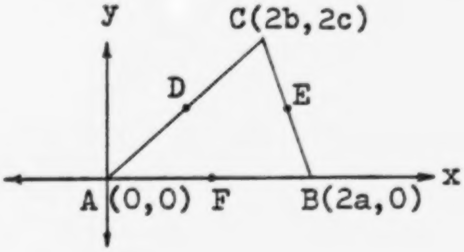
Since the midpoints of $\overline{AC}$ and $\overline{BD}$ have the same coordinates, they are the same point. If two segments intersect at the midpoint of each, they bisect each other.

EXERCISE SET 20.7

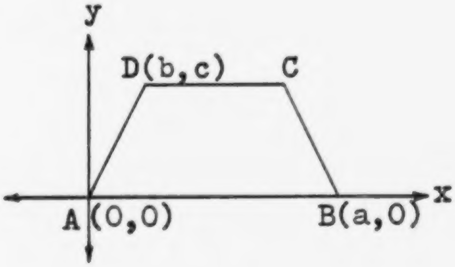
1. Given that ΔABC is equilateral, write both coordinates of C in terms of a.


y
C
x
A (0,0)
B(2a,0)

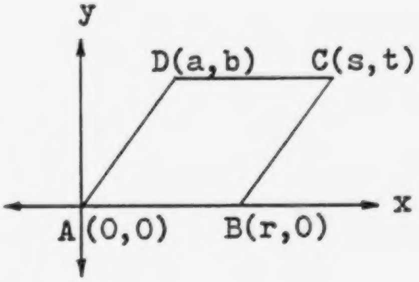
2. Given that D, E, and F are midpoints of $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$, write their coordinates in terms of a, b, and c.


y
C(2b,2c)
D
E
x
A (0,0)
F
B(2a,0)

3. ABCD is an isosceles trapezoid. Express the coordinates of C in terms of a, b, and c.

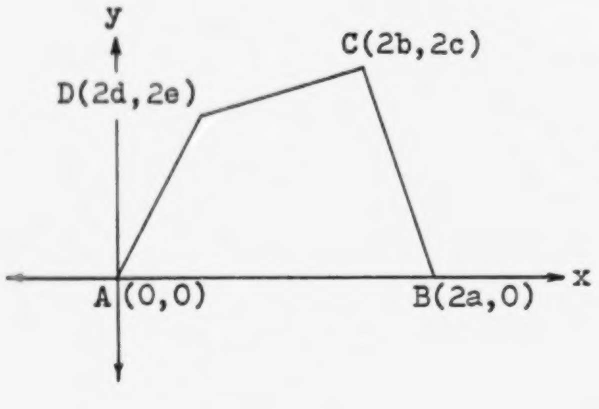

y
D(b,c)
C
x
A (0,0)
B(a,0)

4. ABCD is a rhombus. Show that $r = \sqrt{a^2 + b^2}$, $s = a + \sqrt{a^2 + b^2}$, and $t = b$.


y
D(a,b)
C(s,t)
x
A (0,0)
B(r,0)

Prove each of the following by coordinate methods. Show clearly the assignment of the axes and coordinates.

5. The medians from the endpoints of the base of an isosceles triangle are congruent.
6. A segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is half the length of the third side.
7. The triangle formed by joining the midpoints of the sides of an equilateral triangle is an equilateral triangle.
8. The diagonals of a parallelogram bisect each other.
9. The diagonals of an isosceles trapezoid are congruent.
10. The diagonals of a rhombus are perpendicular. (Hint: If two numbers are negative reciprocals, their product is -1. Use the results of exercise 4 to assign coordinates and show that the product of the slopes of the diagonals is -1.)
11. The figure formed by joining consecutive midpoints of the sides of a rectangle is a rhombus.
12. The figure formed by joining consecutive midpoints of the sides of a quadrilateral is a parallelogram. (Suggestion: Assign the axes and coordinates as shown below.)



Lines and Equations

A brief review of some algebra concepts related to solutions of equations is in order.

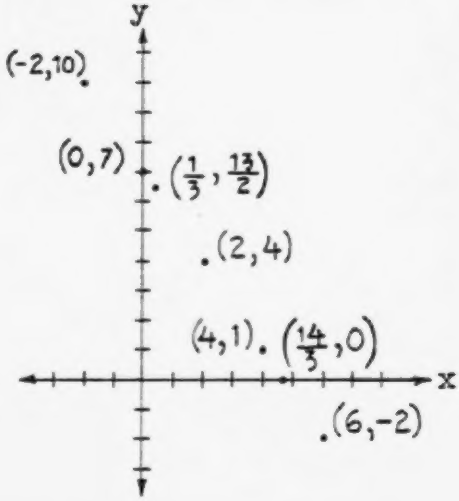
Given the equation $3x + 2 = 14$, we say the solution is 4, because there is no other replacement for x that will yield a true statement.

Given the equation $3x + 2y = 14$, we say a solution is $(4,1)$ as an ordered pair (x,y). But the ordered pairs $(2,4)$, $(0,7)$, $(6,-2)$, $(-2,10)$, $(\frac{14}{3},0)$, $(\frac{1}{3},\frac{13}{2})$ are all solutions. Verify this! In fact, for any real number that we select for x, there is a corresponding real number for y such that (x,y) is a solution. We would have to condlude that the solution set is an infinite set of ordered pairs. The best we can do to write the solution set is to use the following set notation.

$S = \{(x,y) \mid 3x + 2y = 14\}$ which is read as

The solution set, S, is the set of all ordered pairs (x,y) such that $3x + 2y = 14$.

Of course, we don't know much more about it than we did to start with. But if we graph some of the ordered pairs that we do know, we notice that they appear to be collinear. This valuable property is characteristic of all such equations. The following general statement can be proved. We will not prove it here, but we will make very good use of it.



The graph of the solution set of an equation that can be written in the form $ax + by = c$ with a and b not both 0 is a line.

We generally shorten the first part, "The graph of an equation ...

The statement means two things.

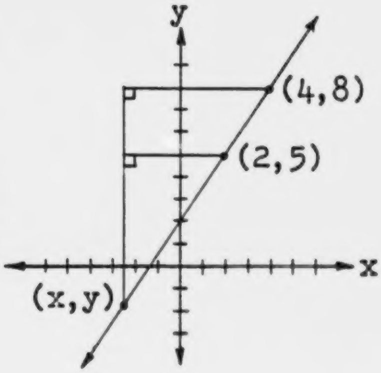
1. Every ordered pair in the solution set has its graph on the line.
2. Every point on the line is the graph of an ordered pair in the solution set.

Our objectives now are the following.

1. Given such an equation, draw its graph.
2. Given conditions that determine a line in the coordinate plane, find an equation for which it is the graph.

We find it easier to approach the second objective first.

Since two points determine a line, we want to find an equation whose graph is the line determined by (2,5) and (4,8). What we want is a statement that is true for the infinite set of ordered pairs (x,y) whose graphs are on the line.



The slope of the line is $m = \frac{8-5}{4-2} = \frac{3}{2}$.

Since the slope of a line is constant, we may use any two points to express it. If (x,y) is any other point, then

$$\frac{y-5}{x-2} = \frac{3}{2}$$

This statement is true for every ordered pair on the line except one. It is not true for (2,5) since $\frac{5-5}{2-2} = \frac{3}{2}$ is false. In fact, the left member, $\frac{0}{0}$, is meaningless. We can eliminate this weakness by multiplying both sides of the equation by $x - 2$.

$$\frac{y-5}{x-2} = \frac{3}{2} \longrightarrow y - 5 = \frac{3}{2}(x-2)$$

Now the ordered pair (2,5) yields $5 - 5 = \frac{3}{2}(2-2)$ which is true. We now know that every ordered pair on the line is a solution of the equation. But how do we know that every solution of the equation is on the line? First, a definition.

Equivalent equations are equations with the same solution set.

The properties of equality allow us to write the following equivalent equations.

$$y - 5 = \frac{3}{2}(x - 2)$$
$$2(y - 5) = 3(x - 2)$$
$$2y - 10 = 3x - 6$$
$$-3x + 2y = 4$$

Since $-3x + 2y = 4$ is of the form $ax + by = c$, its graph is a line, and its solution set is the same as that for $y - 5 = \frac{3}{2}(x - 2)$. This answers the earlier question.

If we had used $(4,8)$ instead of $(2,5)$ the resulting equation would also have been equivalent to $-3x + 2y = 4$. We refer to an equation of the line since there is more than one form. If one point and the slope had been given initially, we can see that this determines a line.

We will generalize this example as a theorem. We use subscripts (x_1, y_1) to denote fixed points and (x,y) as a variable point representative of the infinite set of points on the line. Do not confuse the two.

Theorem 20-7: The equation of the line containing the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

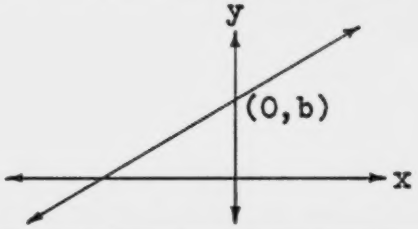
This is called the point-slope form of an equation of a line. The proof would follow the same reasoning as the example using (x_1, y_1) instead of $(2,5)$ and m instead of $\frac{3}{2}$.

The theorem may be used when the information given is either one point and the slope, or two points as in the example.

Another useful equation form is obtained from the point-slope form.

Let m be the slope of a line containing the point $(0,b)$. By the point-slope formula

$$y - b = m(x - 0)$$
$$y = mx + b$$



Theorem 20-8: The equation of the line with slope m and containing the point $(0,b)$ is $y = mx + b$.

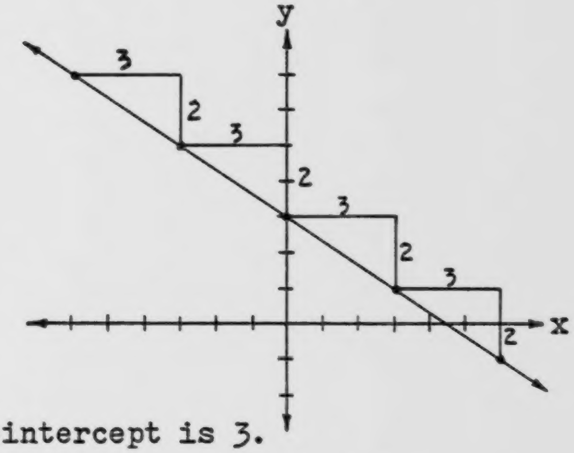
The number b is called the y-intercept and the equation is called the slope-intercept form. It is most useful in connection with the first objective of this section as illustrated next.

Given the equation $2x + 3y = 9$, find its graph.

One method is to find random ordered pairs as we did earlier. This is usually done in a table of values form. Select a value for one variable and solve for the other. Two points will determine the line, but three points are recommended as a check.

A method frequently used is to write an equivalent equation in the slope-intercept form.

$$2x + 3y = 9$$
$$3y = -2x + 9$$
$$y = -\frac{2}{3}x + 3$$



The slope is $-\frac{2}{3}$ and the y-intercept is 3.

We graph the point (0,3) and locate as many other points from that one as we wish by using the slope and counting two vertical and three horizontal in the negative direction.

EXERCISE SET 20.8

For each equation, identify one point and the slope and draw its graph.

1. $y - 3 = \frac{1}{2}(x - 2)$
2. $y + 1 = -3(x - 5)$
3. $y = -\frac{4}{3}(x + 3)$
4. $y + 2 = 4x$

For each equation, identify the slope and y-intercept and draw its graph.

5. $y = \frac{3}{2}x + 1$
6. $y = -3x - 2$
7. $y = -\frac{5}{2}x + 3$
8. $y = \frac{3}{4}x - \frac{1}{4}$

For each equation write an equivalent equation in the slope-intercept form and draw its graph.

9. $2x + 3y = 6$
10. $5x - y = 2$
11. $x + y = 4$
12. $6x - 2y = -8$
13. $x - y = 0$
14. $3x + 4y = 6$

For each equation write an equivalent equation in the form $ax + by = c$ where a, b, and c are integers.

15. $y + 4 = \frac{2}{5}(x - 1)$
16. $y - 2 = -\frac{1}{2}x$
17. $y + 1 = -4(x + 3)$
18. $y = -\frac{9}{4}(x - 2)$

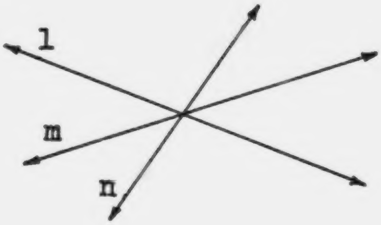
Each of the following conditions determines a line. Write an equation of the line in the form $ax + by = c$ where a, b, and c are integers.

19. Contains points (2,3) and (5,4)
20. Contains points (3,-1) and (-2,4)
21. Contains points (0,-4) and (4,2)
22. Contains points (-2,5) and (6,5)
23. Contains points (-4,-3) and (-1,5)
24. Contains points (0,0) and (4,-2)
25. Contains points (-3,4) and (-3,-3)
26. Contains points (10,3) and (-8,4)
27. Contains point (2,1) with slope $\frac{3}{4}$
28. Contains point (-3,6) with slope -4
29. Contains point (4,-5) and parallel to the line whose equation is $y = 3x + 2$
30. $\triangle ABC$ has vertices at points A(7,0), B(2,2), and C(-1,-4). Write the equation of the line containing the median from B to $\overline{AC}$.
31. $\triangle RST$ has vertices at points R(-2,-1), S(-4,6), and T(-5,1). Write the equation of the line containing the altitude from T to $\overline{RS}$.
32. Given the points E(6,2) and F(2,5), write the equation of the perpendicular bisector of $\overline{EF}$.

Concurrence Theorems

Three or more lines are concurrent if there is exactly one point common to all the lines.

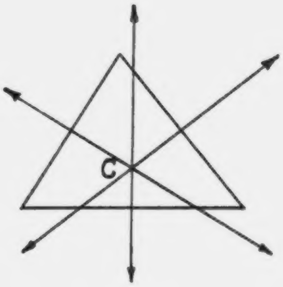
l, m, and n are concurrent lines



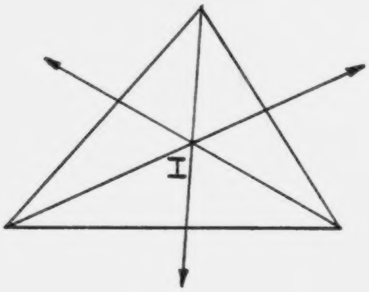
Segments or rays are concurrent by the same definition.

We have already considered two cases of concurrency as illustrated below.

To circumscribe a circle about a triangle, we showed that the perpendicular bisectors of the three sides are concurrent at a point equidistant from the three vertices called the circumcenter.



To inscribe a circle in a triangle, we showed that the three angle bisectors of a triangle are concurrent at a point equidistant from the three sides called the incenter.

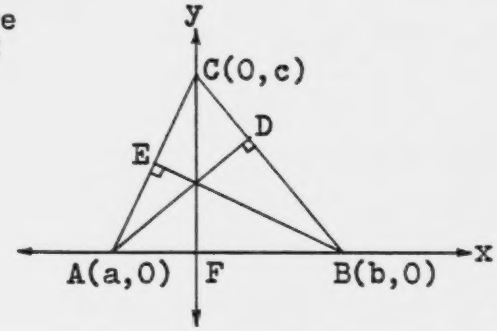


Two more cases involving concurrent lines related to triangles are significant. We will use coordinate methods to prove the concurrency to help illustrate the versatility of this approach.

Theorem 20-9: The lines containing the altitudes of a triangle are concurrent.

Given: $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are altitudes of $\triangle ABC$

Prove: $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent



Argument

Let $\overline{AB}$ and $\overline{CF}$ lie in the x and y axes since $\overline{CF} \perp \overline{AB}$. Assign coordinates as shown.

$$\left.\begin{array}{r}\left.\begin{array}{r}\text{①}\ m(\overline{BC})=\frac{c-0}{0-b}=-\frac{c}{b}\\ \text{②}\ \overline{AD}\perp\overline{BC}\end{array}\right\}\xrightarrow{\text{③}} m(\overleftrightarrow{AD})=\frac{b}{c}\\ \text{④}\ \overleftrightarrow{AD}\text{ contains }(a,0)\end{array}\right\}\xrightarrow{\text{⑤}}\text{equation of }\overleftrightarrow{AD}\text{ is } y=\frac{b}{c}(x-a)$$

The intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{CF}$ (y-axis) must have abscissa 0. Substituting 0 for x in the equation of $\overleftrightarrow{AD}$,

$$y=\frac{b}{c}(0-a)=\frac{b}{c}(-a)=-\frac{ab}{c}$$

So the point $\left(0,-\frac{ab}{c}\right)$ is the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{CF}$.

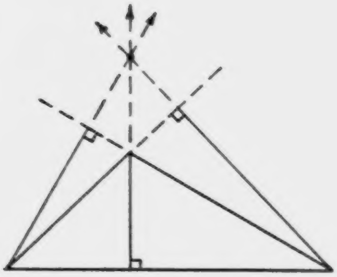
$$\left.\begin{array}{r}\left.\begin{array}{r}\text{⑥}\ m(\overline{AC})=\frac{c-0}{0-a}=-\frac{c}{a}\\ \text{⑦}\ \overline{BE}\perp\overline{AC}\end{array}\right\}\xrightarrow{\text{⑧}} m(\overleftrightarrow{BE})=\frac{a}{c}\\ \text{⑨}\ \overleftrightarrow{BE}\text{ contains }(b,0)\end{array}\right\}\xrightarrow{\text{⑩}}\text{equation of }\overleftrightarrow{BE}\text{ is } y=\frac{a}{c}(x-b)$$

The intersection of $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ also has abscissa 0.

$$y=\frac{a}{c}(0-b)=\frac{a}{c}(-b)=-\frac{ab}{c}$$

So the point $\left(0,-\frac{ab}{c}\right)$ is the intersection of $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$.

$\therefore$ $\left(0,-\frac{ab}{c}\right)$ is common to $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ and they are concurrent.

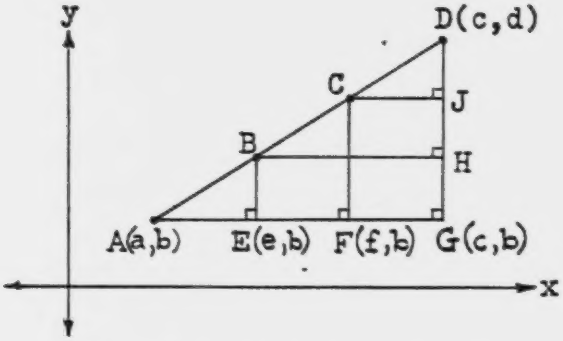


The theorem must refer to the lines containing the altitudes because if the triangle were obtuse, the altitudes as segments would not intersect at all. But the lines containing them would be concurrent in a point in the exterior of the triangle. Our proof is just as valid for this case.

The point of intersection of the altitudes of a triangle is the orthocenter of the triangle.

What is the point of concurrency of the altitudes of a right triangle?

Before considering the next theorem, it will be helpful to establish the coordinates of the trisection points of a segment. The development is similar to that for the midpoint formula.



Let AB = BC = CD and draw the ⊥s shown. Then AE = EF = FG. Let the coordinates of E and F be (e,b) and (f,b).

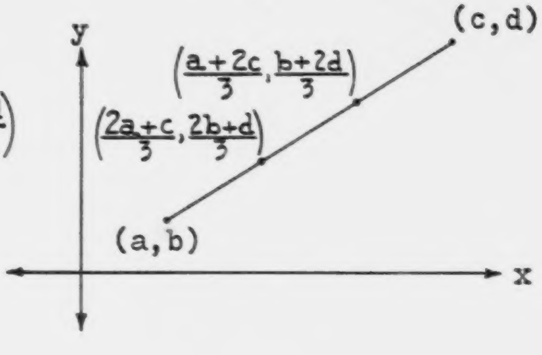
$$\left.\begin{array}{l} AE = e - a \\ EG = c - e \\ 2\cdot AE = EG \end{array}\right\} \rightarrow 2\cdot(e-a) = c - e \rightarrow 2e - 2a = c - e \rightarrow 3e = 2a + c \rightarrow e = \frac{2a+c}{3}$$

$$\left.\begin{array}{l} AF = f - a \\ FG = c - f \\ AF = 2\cdot FG \end{array}\right\} \rightarrow f - a = 2\cdot(c-f) \rightarrow f - a = 2c - 2f \rightarrow 3f = a + 2c \rightarrow f = \frac{a+2c}{3}$$

Similarly, the ordinates of J and H are $\frac{2b+d}{3}$ and $\frac{b+2d}{3}$

So the coordinates of B and C are

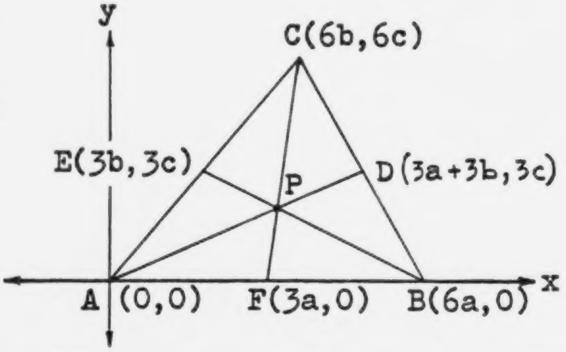
$$\left(\frac{2a+c}{3}, \frac{2b+d}{3}\right) \text{ and } \left(\frac{a+2c}{3}, \frac{b+2d}{3}\right)$$


y
(c,d)
$\left(\frac{a+2c}{3}, \frac{b+2d}{3}\right)$
$\left(\frac{2a+c}{3}, \frac{2b+d}{3}\right)$
(a,b)
x

Theorem 20-10: The medians of a triangle are concurrent at a point which is two thirds the distance from each vertex to the midpoint of the opposite side.

Given: $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are medians of ΔABC

Prove: $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at some point P such that $AP = \frac{2}{3}AD$, $BP = \frac{2}{3}BE$, $CP = \frac{2}{3}CF$



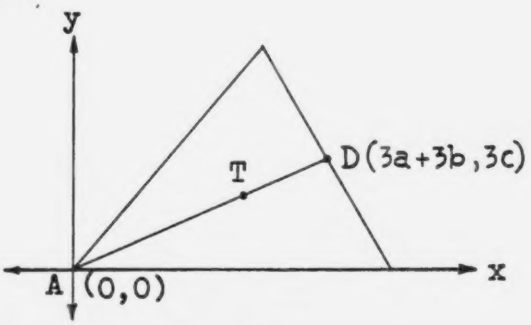
Argument

Assign the placement and coordinates as shown. (They are chosen so the numerators of the fractions below will be divisible by 3.)

Let T be a trisection point of $\overline{AD}$ such that $AT = \frac{2}{3}AD$.

The coordinates of T are $\left(\frac{0 + 2(3a + 3b)}{3}, \frac{0 + 2(3c)}{3}\right)$

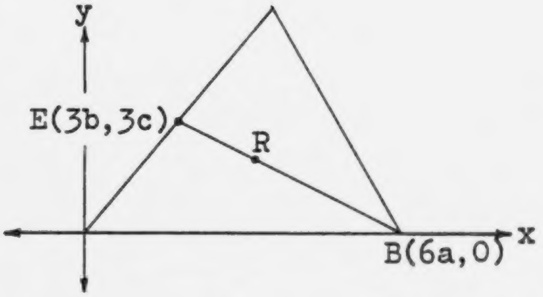
or $(2a + 2b, 2c)$



Let R be a trisection point of $\overline{BE}$ such that $BR = \frac{2}{3}BE$.

The coordinates of R are $\left(\frac{6a + 2(3b)}{3}, \frac{0 + 2(3c)}{3}\right)$

or $(2a + 2b, 2c)$



Similarly, a trisection point of $\overline{CF}$ would have coordinates $(2a + 2b, 2c)$.

$\therefore$ $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ all have this point in common which is two thirds the distance from each vertex to the midpoint of the opposite side.

EXERCISE SET 20.9

1. $\triangle ABC$ has vertices at points A(2,1), B(9,1), and C(5,7). Since $\overline{AB}$ is horizontal, the altitude from C to $\overline{AB}$ is vertical and the equation of the line containing it is $x = 5$.
   a) Write the equation of the line containing the altitude from B.
   b) Substitute 5 for x in the equation of (a) to find the coordinates of the intersection of the altitudes from C and B.
   c) Write the equation of the line containing the altitude from A.
   d) Substitute the coordinates of (b) into the equation of (c) to show that the three altitudes are concurrent at that point.

2. $\triangle DEF$ has vertices at points D(12,4), E(4,6), and F(2,2).
   a) Write the equation of the line containing the median from D.
   b) Write the equation of the line containing the median from E.
   c) Find the coordinates of the intersection of the medians from D and E.
   d) Write the equation of the line containing the median from F.
   e) Substitute the coordinates of (c) into the equation of (d) to show that the three medians are concurrent at that point.
   f) Show that the intersection of the three medians is two thirds of the way from each vertex to the midpoint of the opposite side.

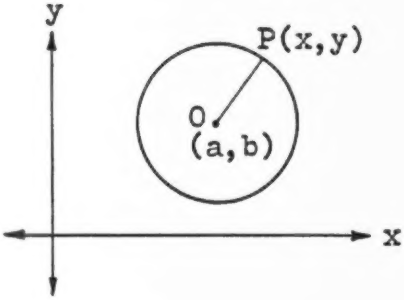
3. $\triangle GHJ$ has vertices at points G(1,2), H(13,2), and J(5,10).
   a) Write the equation of the perpendicular bisector of $\overline{GH}$.
   b) Write the equation of the perpendicular bisector of $\overline{HJ}$.
   c) Find the coordinates of the intersection of the perpendicular bisectors of $\overline{GH}$ and $\overline{HJ}$.
   d) Write the equation of the perpendicular bisector of $\overline{GJ}$.
   e) Substitute the coordinates of (c) into the equation of (d) to show that the three perpendicular bisectors are concurrent at that point.
   f) Show that the point of intersection of the three perpendicular bisectors is equidistant from the three vertices.

4. ΔKLM has vertices at points K(-5,-4), L(9,-4), and M(5,8).
   a) Find the coordinates of the intersection of the altitudes as in exercise 1.
   b) Find the coordinates of the intersection of the medians by finding the trisection point of one median. If (a,b) is a vertex and (c,d) is the midpoint of the opposite side, the desired point is $\left(\frac{a+2c}{3}, \frac{b+2d}{3}\right)$.
   c) Find the coordinates of the intersection of the perpendicular bisectors as in exercise 3.
   d) Show that the three points of parts (a), (b), and (c) are collinear by showing that the lines containing each pair of them have the same slope.
   e) The intersection of the medians is called the centroid of the triangle. It was stated earlier that the intersection of the altitudes is the orthocenter and the intersection of the perpendicular bisectors is the circumcenter of the triangle. Show that the distance from the centroid to the orthocenter is twice the distance from the centroid to the circumcenter.

It has been proved that the centroid, orthocenter, and circumcenter of any triangle are always collinear. The line is called the Euler (Oiler) line of the triangle. It has also been proved that the distance from the centroid to the orthocenter is always twice the distance from the centroid to the circumcenter.

Circles in the Coordinate Plane

Let O be the center of a circle (a fixed point) with coordinates (a,b), and let P be any point on the circle (a variable point) with coordinates (x,y). Since a point is on a circle if and only if it is at a distance from the center equal to the radius, we can use the distance formula to describe the set of all points, P, on the circle.



$$\sqrt{(x-a)^2+(y-b)^2}=r \longrightarrow (x-a)^2+(y-b)^2=r^2$$

Then $S=\{(x,y)\,|\,(x-a)^2+(y-b)^2=r^2\}$ is the set of all ordered pairs associated with the points of the circle. That is, the circle is the graph of the equation.

Theorem 20-11: The graph of the solution set of the equation $(x-a)^2+(y-b)^2=r^2$ is a circle with center (a,b) and radius r.

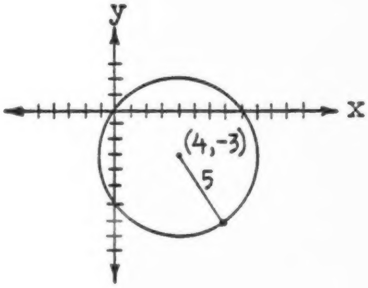
Just as with the equation of a line, we have two objectives.

1. Given an equation that can be written in this form, to draw the circle.
2. Given information that determines a circle in the coordinate plane, to write its equation.

The first is simple when the equation is in the desired form.

Example: Graph the equation $(x-4)^2+(y+3)^2=25$.

Since $a=4$, $b=-3$, and $r=5$, the graph is a circle with center (4,-3) and radius 5.



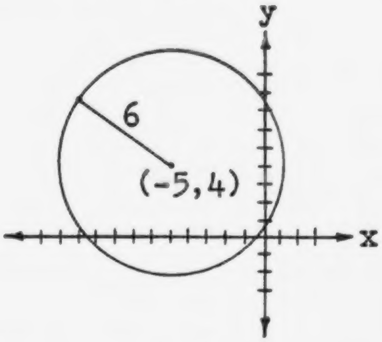
The second objective is equally simple when the information given is the center and the radius.

Example: Write the equation of the circle with center at $(-5,4)$ and radius 6.

The equation is

$$(x-(-5))^2+(y-4)^2=6^2$$

or $(x+5)^2+(y-4)^2=36$



Both objectives have less simple variations. As an example in objective (1), the graph of the equation

$$x^2+y^2-6x+14y+42=0$$

is a circle, but we can't tell anything about the circle from this form. The trick is to write an equivalent equation in the desired form.

Recall the nature of the square of a binomial as a trinomial. $(x+\underline{a})^2=x^2+\underline{2a}x+\underline{a^2}$

Notice the three underlined numbers are all in terms of a. If we know any one of these numbers, we can find both the other two. We write our original equation above in a form that will allow us to make use of this relationship.

$$x^2-6x \qquad +y^2+14y \qquad =-42$$

Now we can "complete the square" on x and y by considering the -6 and 14 as 2a above in separate trinomials and using the addition property of equality.

$$x^2-6x+9+y^2+14y+49=-42+9+49$$

$$\tfrac{1}{2}\cdot 6 \searrow \nearrow 3^2 \qquad \tfrac{1}{2}\cdot 14 \searrow \nearrow 7^2$$

$$(x-3)^2 \quad + \quad (y+7)^2 \quad =16$$

Now we can identify the graph of the equivalent equations as a circle with center $(3,-7)$ and radius 4.

EXERCISE SET 20.10

Find the coordinates of the center and the radius of the circle whose equation is as follows.

1. $(x-3)^2+(y-7)^2=9$
2. $(x+2)^2+(y-1)^2=1$
3. $(x+4)^2+(y+10)^2=12$
4. $(x-5)^2+y^2=36$
5. $x^2+(y+3)^2=6$
6. $x^2+y^2=100$
7. Given the equation $(x-3)^2+(y-5)^2=25$ and its graph. For each value of x determine whether there are two points, one point, or no points on the graph and find the corresponding value of y at each point if it exists.

   a) $x=-3$
   b) $x=-2$
   c) $x=0$
   d) $x=3$
   e) $x=5$
   f) $x=6$
   g) $x=8$
   h) $x=10$

Write the equation of the circle determined by each of the following conditions.

8. Center at (1,4) and radius is 2.
9. Center at (-3,5) and radius is 7.
10. Center at (0,0) and radius is 1.
11. Center at (-4,0) and radius is $\frac{1}{2}$.
12. Center at (0,7) and radius is $\sqrt{3}$.
13. Center at (-1,-3) and radius is $5\sqrt{2}$.
14. Center at (5,4) and tangent to the x-axis.
15. Center at (0,0) and contains the point (4,-3).
16. Center at (-2,1) and contains the point (3,5).
17. The endpoints of a diameter are (2,-3) and (2,9).
18. The endpoints of a diameter are (5,2) and (-5,-2).
19. Three noncollinear points determine a circle. Find the equation of the circle containing the points (1,2), (7,2), and (6,7).

Find the coordinates of the center and the radius of the circle whose equation is as follows.

20. $x^2+y^2+8x+4y=5$
21. $x^2+y^2-2x+10y-10=0$
22. $x^2+y^2+16y+28=0$
23. $x^2-12x+y^2=4$

CHAPTER 21 - AREAS OF POLYGONS AND CIRCLES

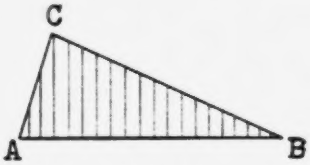
or IT'S WHAT'S INSIDE THAT COUNTS

Definition of Area

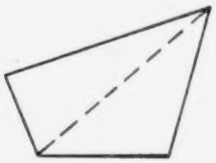
Do you know the formula for the area of a triangle? How can you when a triangle doesn't have an area? A triangle is the union of three segments and a segment does not have an area. Of course, we know what we mean when we speak of the area of a triangle. But in the true spirit of our course, we will be precise with our language at first and take liberties with it later.

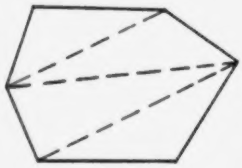
A triangular region is the union of a triangle and its interior.

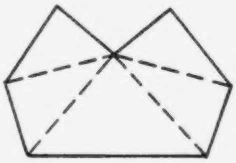
Triangular region ABC is said to be bounded by ΔABC.

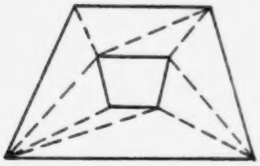


A polygonal region is the union of a finite number of triangular regions all in the same plane, such that if two of these intersect, their intersection is either a point or a segment.





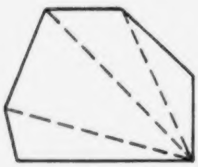


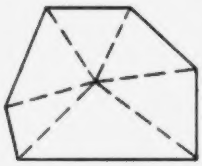


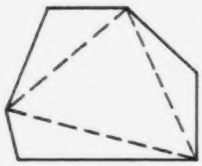
The definition allows a polygonal region, such as the one on the right, to have a "hole" in it.

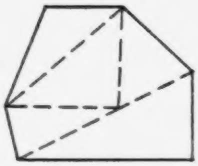
Each of the polygonal regions above is said to be dissected into the triangular regions shown.

Any polygonal region can be dissected into triangular regions in an unlimited number of ways as illustrated below.









Now we can talk about "area". Our approach will be similar to that used throughout the course; that is, to establish definitions, postulates and theorems. Some of the proofs will not be highly rigorous, as most of the exercises will require you to make applications of the formulas derived in the theorems.

Postulate 24: To every polygonal region there corresponds a unique positive real number.

The area of a polygonal region is the number assigned to it by postulate 24. The area of region R will be denoted A(R).

Notice that area is defined as a number, not as a surface.

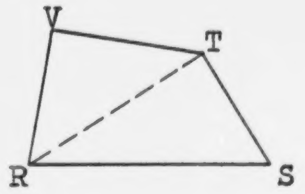
Postulate 25: If two triangles are congruent, then the regions bounded by them have the same area.

Postulate 26: If a polygonal region is dissected into n polygonal regions, then its area is the sum of the areas of those n regions.

The last postulate allows us to add areas in a manner similar to the addition of measures of segments, angles, and arcs.

We will now agree to shorten the phrase "area of triangular region ABC" to "area of ΔABC" and write $A(\Delta ABC)$.

$$A(RSTV) = A(\Delta RST) + A(\Delta RVT)$$

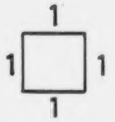


Areas of Rectangles and Triangles

It is possible that each of the regions below could correspond to the same positive real number; that is, have the same area.

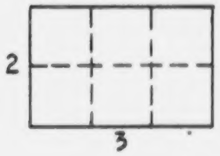
Then we could associate any of these with the number 1, and express the area of any other region in terms of this "unit". Something tells us that the square is the most reasonable choice for this purpose.

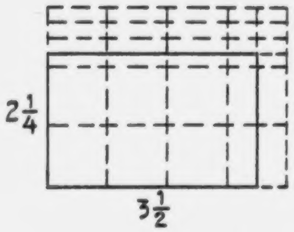
The unit of area is defined as a square each of whose sides is 1 unit of length. It is 1 square unit.

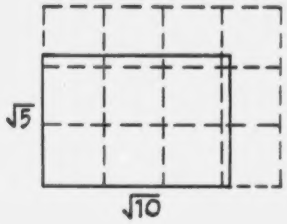


In applications to physical problems, if the unit of length is 1 inch, the unit of area is 1 square inch, and likewise for the other common units.

You are already familiar with most of the basic area formulas. Probably the formula that is most easily acceptable without proof is that for a rectangle. The measure of any side of a rectangle is its length, and the measure of either adjacent side is the width. If the length and width are both whole numbers, we can find the area by counting square units as in the figure on the left.







$A = 3 \cdot 2 = 6$

$A = 3\frac{1}{2} \cdot 2\frac{1}{4} = \frac{7}{2} \cdot \frac{9}{4} = \frac{63}{8} = 7\frac{7}{8}$

$A \doteq 3.16 \cdot 2.24 \doteq 7.08$

If the length and width involve fractions as in the middle figure, it is more awkward, but we can still count and add the fractional parts of a square unit. If the length and width are irrational as in the figure on the right, we can only approximate the area by counting. In the first two cases, we can confirm without question that the area obtained by counting is the same as the number obtained by multiplying length times width. In the third case, it seems reasonable that this is also true.

Postulate 27: The area of a rectangle is the product of its length and its width. $A = lw$

Having accepted the postulate, we now say the area of the third rectangle in our previous example is $\sqrt{10} \cdot \sqrt{5} = \sqrt{50} = 5\sqrt{2}$ in simplest form.

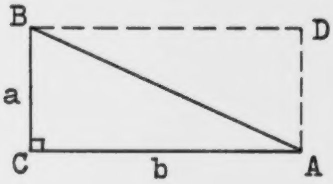
Corollary 27a: The area of a square is the square of the length of its side. $A = s^2$

The remaining area formulas for polygons can now be deduced. We will devote the remainder of this section to triangles. Any references to base and altitude hereafter will mean either the segments or their lengths. The context in which it is used will indicate clearly which is being named.

Theorem 21-1: The area of a right triangle is half the product of its legs.

Given: rt. $\triangle ABC$ with legs a and b

Prove: $A = \frac{1}{2}ab$



Argument

① Draw $\overline{BD}$ and $\overline{AD}$ such that BCAD is a rectangle.

$$\ldots \xrightarrow{\textcircled{2}} \triangle ABC \cong \triangle BAD \xrightarrow{\textcircled{3}} \left.\begin{array}{r} A(\triangle ABC) = A(\triangle BAD) \\ \textcircled{4}\ A(\triangle ABC) + A(\triangle BAD) = A(BCAD) \end{array}\right\} \rightarrow \ldots \xrightarrow{\textcircled{5}}$$

$$\xrightarrow{\textcircled{5}} 2 \cdot A(\triangle ABC) = A(BCAD) \xrightarrow{\textcircled{6}} \left.\begin{array}{r} A(\triangle ABC) = \frac{1}{2} \cdot A(BCAD) \\ \textcircled{7}\ A(BCAD) = ab \end{array}\right\} \xrightarrow{\textcircled{8}}$$

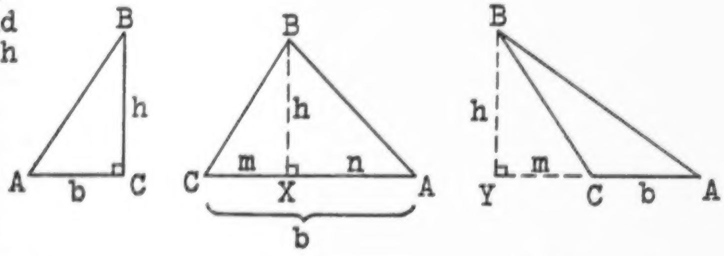
$$\xrightarrow{\textcircled{8}} A(\triangle ABC) = \frac{1}{2}ab$$

$\therefore$ The formula for area of any rt. $\triangle$ with legs a and b is $A = \frac{1}{2}ab$.

Theorem 21-2: The area of a triangle is half the product of any base and the altitude to that base.

Given: ΔABC with base b and altitude h

Prove: $A = \frac{1}{2}bh$



Case I Case II Case III

Argument

Case I is theorem 21-1.

Case II

$$\left.\begin{array}{r} \text{①}\ A(\Delta BCX) = \frac{1}{2}mh \\ \text{②}\ A(\Delta BAX) = \frac{1}{2}nh \\ \text{③}\ A(\Delta ABC) = A(\Delta BCX) + A(\Delta BAX) \end{array}\right\} \xrightarrow{\text{④}} \left.\begin{array}{r} A(\Delta ABC) = \frac{1}{2}mh + \frac{1}{2}nh = \frac{1}{2}(m+n)h \\ \text{⑤}\ m+n=b \end{array}\right\} \xrightarrow{\text{⑥}}$$

$$\xrightarrow{\text{⑥}} A(\Delta ABC) = \frac{1}{2}bh$$

Case III

$$\left.\begin{array}{r} \text{⑦}\ A(\Delta ABY) = \frac{1}{2}(m+b)h = \frac{1}{2}mh + \frac{1}{2}bh \\ \text{⑧}\ A(\Delta CBY) = \frac{1}{2}mh \\ \text{⑨}\ A(\Delta ABC) = A(\Delta ABY) - A(\Delta CBY) \end{array}\right\} \xrightarrow{\text{⑩}}$$

$$\xrightarrow{\text{⑩}} A(\Delta ABC) = \left(\frac{1}{2}mh + \frac{1}{2}bh\right) - \frac{1}{2}mh = \frac{1}{2}bh$$

$\therefore$ The formula for area of any triangle with base b and altitude to that base h, is $A = \frac{1}{2}bh$.

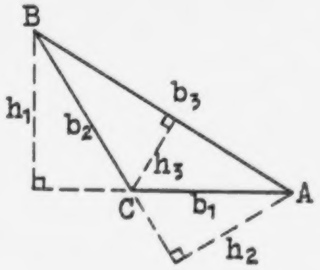
Since any side of a triangle can be considered a base and each base has its corresponding altitude, the formula can be written three ways for any triangle.

$$A = \frac{1}{2} b_1 h_1$$

$$A = \frac{1}{2} b_2 h_2$$

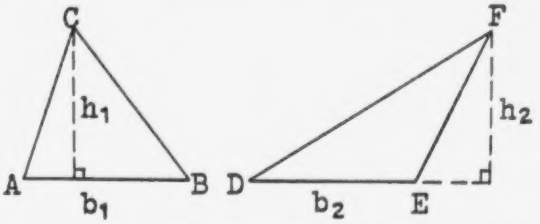
$$A = \frac{1}{2} b_3 h_3$$

$$\therefore\ A = \frac{1}{2} b_1 h_1 = \frac{1}{2} b_2 h_2 = \frac{1}{2} b_3 h_3$$



Corollary 21-2a: If a side of one triangle is congruent to a side of another triangle and the corresponding altitudes to those sides are congruent, they have equal areas.

$$\left.\begin{array}{l} b_1 = b_2 \\ h_1 = h_2 \end{array}\right\} \rightarrow A(\Delta ABC) = A(\Delta DEF)$$

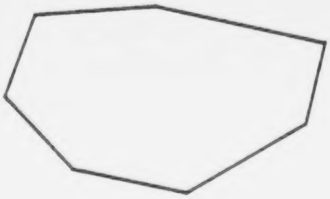


Notice that the corollary does not require the triangles to be congruent.

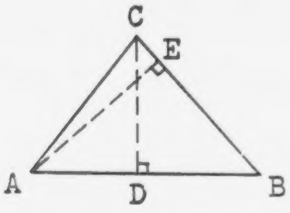
Corollary 21-2b: The area of an equilateral triangle with side of length s is expressed by the formula $A = \frac{s^2\sqrt{3}}{4}$.

EXERCISE SET 21.1

1. Copy the figure three times and show three different dissections into triangular regions.



2. By postulate 25, $\Delta ABC \cong \Delta PQR \longrightarrow$?
3. Given pentagon EFGHJ. If all diagonals are drawn from vertex G, then A(EFGHJ) = ?
4. Find the area of each rectangle whose length and width are as follows.
   - a) 8, 5
   - b) 4.3, 1.7
   - c) $6\frac{1}{4}$ in., $2\frac{1}{2}$ in.
   - d) $\sqrt{8}$ ft., $\sqrt{2}$ ft.
   - e) 450 cm., 35 cm.
   - f) 10 ft. 3 in., 8 ft. 8 in.
5. The area of a square and a rectangle are equal. If the dimensions of the rectangle are 10 ft. by 5 ft., find the side of the square.
6. The perimeter of a square is 36. Find its area.
7. Find the area of each triangle whose base and altitude are as follows.
   - a) 14,9
   - b) 36 mm., 54 mm.
   - c) $7\frac{1}{2}$ in., $10\frac{1}{2}$ in.
   - d) 4 ft. 9 in., 1 ft. 6 in.
8. If AB = 20, CD = 12, AE = 15, find BC.



9. The side of a square has length s. How is the area of the square changed if s is doubled? tripled? divided by 2?

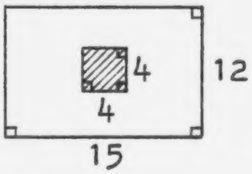
10. Find the area of an equilateral triangle with side 10.

11. Find the area of an equilateral triangle with altitude 12.

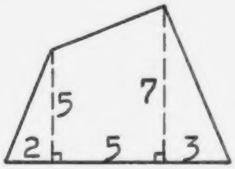
12. Find the area of an isosceles right triangle with hypotenuse 8.

Find the areas of each of the following polygonal regions. Do not include the shaded region.

13.



14.



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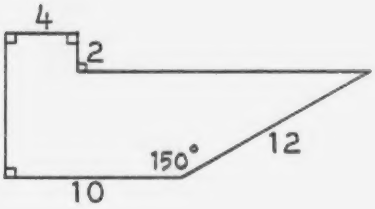
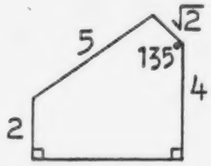
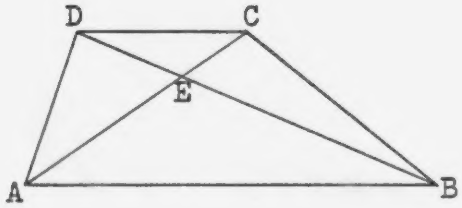


16.



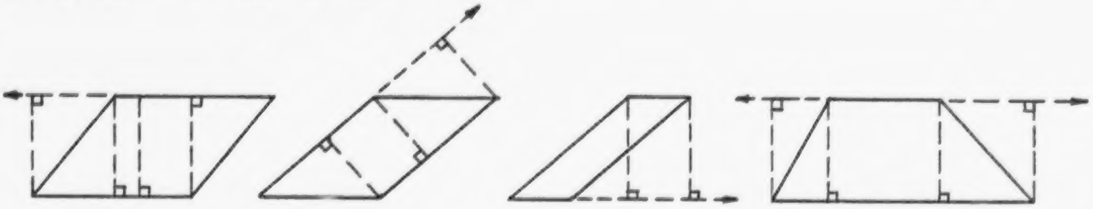
17. Prove: A median of a triangle separates the triangle into two triangles of equal area.

18. Given: $\overline{DC} \parallel \overline{AB}$
Prove: $A(\triangle AED) = A(\triangle BEC)$



Areas of Special Quadrilaterals

If a quadrilateral has at least one pair of parallel sides, those sides may be considered as bases. An altitude is any segment with an endpoint in one base and perpendicular to the line containing the opposite base with the other endpoint in that line.

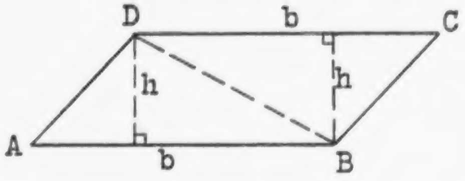


Several altitudes are shown in the above figures. Since parallel lines are everywhere equidistant, all altitudes to the same base are equal in length.

Theorem 21-3: The area of a parallelogram is the product of any base and the corresponding altitude.

Given: ▱ABCD with
base b and
altitude h

Prove: $A = bh$



Argument

① Draw $\overline{BD}$ and altitudes from D and B.

② A(▱ABCD) = $A(\triangle ABD) + A(\triangle CDB)$

③ $A(\triangle ABD) = \frac{1}{2}bh$

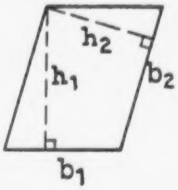
④ $A(\triangle CDB) = \frac{1}{2}bh$

⑤ → A(▱ABCD) = $\frac{1}{2}bh + \frac{1}{2}bh = bh$

∴ The formula for area of any parallelogram with base b and corresponding altitude h, is $A = bh$.

Just as the triangle formula has three possible forms, the parallelogram formula has two possible forms.

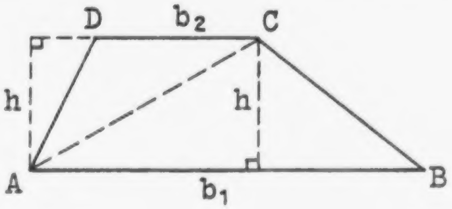
$A = b_1h_1 = b_2h_2$



Theorem 21-4: The area of a trapezoid is half the product of its altitude and the sum of its bases.

Given: trap. ABCD with bases b_1 and b_2 and altitude h

Prove: $A = \frac{1}{2}h(b_1 + b_2)$



Argument

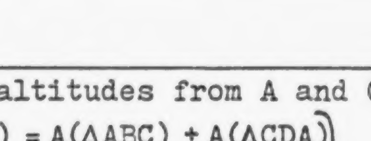
$$\left.\begin{array}{l} \text{①} \text{ Draw } \overline{AC} \text{ and altitudes from A and C.} \\ \text{②} \ A(\text{trap. }ABCD) = A(\triangle ABC) + A(\triangle CDA) \\ \qquad\qquad \text{③} \ A(\triangle ABC) = \frac{1}{2}b_1h \\ \qquad\qquad \text{④} \ A(\triangle CDA) = \frac{1}{2}b_2h \end{array}\right\} \xrightarrow{\text{⑤}}$$

$\xrightarrow{\text{⑤}} A(\text{trap. }ABCD) = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}h(b_1 + b_2)$

∴ The formula for area of a trapezoid with altitude h and bases b_1 and b_2, is $A = \frac{1}{2}h(b_1 + b_2)$.

Corollary 21-4a: The area of a trapezoid is the product of the length of its median and its altitude.

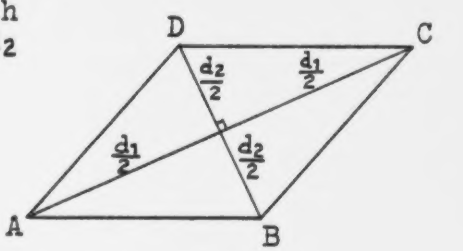
$A = mh$



Theorem 21-5: The area of a rhombus is half the product of the lengths of its diagonals.

Given: rhombus ABCD with $AC = d_1$ and $BD = d_2$

Prove: $A = \frac{d_1 d_2}{2}$



Informal argument

The area of each of the four ≅ Δs is $\frac{1}{2} \cdot \frac{d_1}{2} \cdot \frac{d_2}{2} = \frac{d_1 d_2}{8}$.

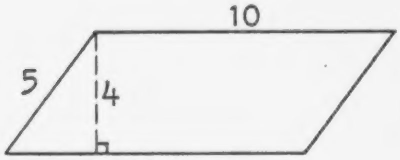
The area of the rhombus is four times the area of each Δ.

$\therefore$ A(rhombus ABCD) $= 4 \cdot \frac{d_1 d_2}{8} = \frac{d_1 d_2}{2}$

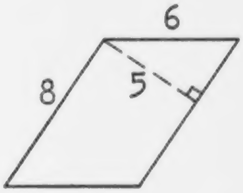
EXERCISE SET 21.2

Find the area of each parallelogram.

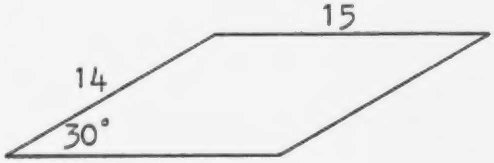
1.



2.



3.



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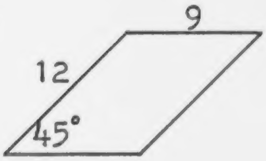
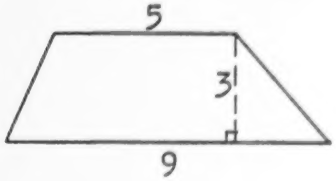


5. The bases of a parallelogram are 8 and 10. If the shorter altitude is 6, find the longer altitude.

6. A parallelogram and a triangle have equal areas and equal bases. How do their altitudes compare?

Find the area of each trapezoid.

7.



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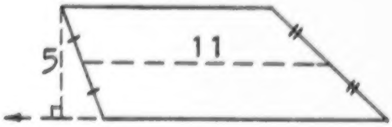

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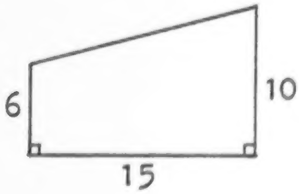
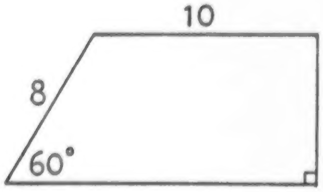


10.



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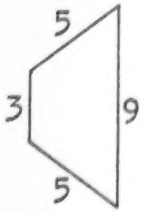

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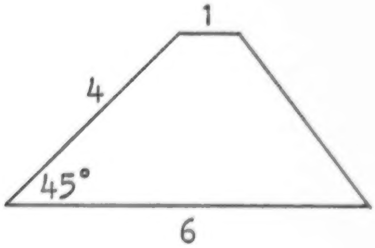


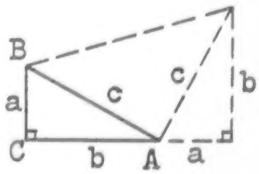
13. The bases of a trapezoid are 14 in. and 10 in. and its area is 84 sq. in. Find the altitude.

14. One base of a trapezoid is twice as long as the other base. If the altitude is 5 ft. and the area is 60 sq. ft., find the lengths of the two bases.

15. The diagonals of a rhombus have lengths 18 cm. and 10 cm. Find its area.

16. The sides of a rhombus have length 15 in. and the length of one diagonal is 18 in. Find the area.

17. The Pythagorean Theorem was proved by James Garfield before he became President of the U.S. His plan was to write the area of the trapezoid as the sum of the areas of the three triangles.
Show that $a^2 + b^2 = c^2$.

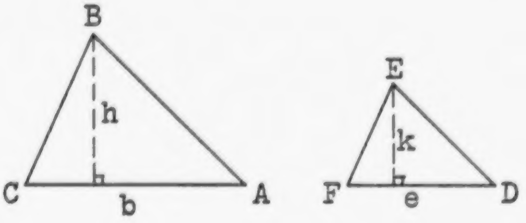


Area Relations of Similar Triangles and Polygons

Theorem 21-6: The ratio of the areas of two similar triangles is the square of the ratio of the lengths of any two corresponding sides.

Given: $\triangle ABC \sim \triangle DEF$

Prove: $\frac{A(\triangle ABC)}{A(\triangle DEF)} = \left(\frac{b}{e}\right)^2$



Argument

① Draw altitude h from B and altitude k from E.

② $\frac{A(\triangle ABC)}{A(\triangle DEF)} = \frac{\frac{1}{2}bh}{\frac{1}{2}ek} = \frac{bh}{ek} = \frac{b}{e}\cdot\frac{h}{k}$

③ $\triangle ABC \sim \triangle DEF \xrightarrow{④} \frac{h}{k} = \frac{b}{e}$

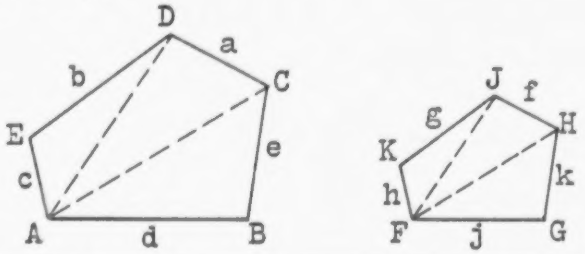
$\xrightarrow{⑤} \frac{A(\triangle ABC)}{A(\triangle DEF)} = \frac{b}{e}\cdot\frac{b}{e} = \frac{b^2}{e^2} = \left(\frac{b}{e}\right)^2$

$\therefore \triangle ABC \sim \triangle DEF \longrightarrow \frac{A(\triangle ABC)}{A(\triangle DEF)} = \left(\frac{b}{e}\right)^2$

Theorem 21-7: The ratio of the areas of two similar polygons is the square of the ratio of the lengths of any two corresponding sides. (shown for $n = 5$)

Given: $ABCDE \sim FGHJK$

Prove: $\frac{A(ABCDE)}{A(FGHJK)} = \left(\frac{a}{f}\right)^2$



The formal proof is rather lengthy, but it follows from the fact that the corresponding pairs of triangles shown can be proved similar. We can use theorem 21-6 and the properties of proportions. The proof would be similar for any n-gon.

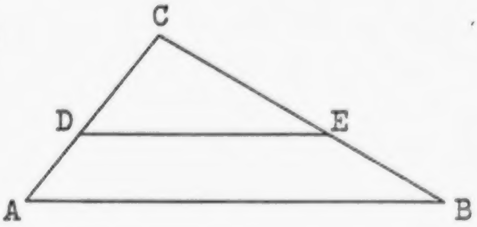
EXERCISE SET 21.3

1. Given $\triangle ABC \sim \triangle DEF$. Find $\frac{A(\triangle ABC)}{A(\triangle DEF)}$.
   - a) $AB = 3$, $DE = 5$
   - b) $BC = 4$, $EF = 1$
   - c) $AC = \frac{2}{3}$, $DF = 2$
   - d) $BC = \sqrt{3}$, $EF = 3$
   - e) $CA = \frac{2}{5} \cdot FD$
   - f) $AB = 6 \cdot DE$

2. Pentagon $ABCDE \sim$ pentagon $FGHJK$. $AB = 7$, $FG = 2$.
   $\frac{A(ABCDE)}{A(FGHJK)} = \ ?$

3. $RSTV \sim WXYZ$. $A(RSTV) = 147$, $A(WXYZ) = 75$, $ST = 7$. Find XY.

4. $\overline{DE} \parallel \overline{AB}$. Find $\frac{A(\triangle ABC)}{A(\triangle DEC)}$ in each of the following.
   - a) $AB = 8$, $DE = 6$
   - b) $CD = 4$, $DA = 3$
   - c) $BE = 2\sqrt{3}$, $EC = 3\sqrt{3}$



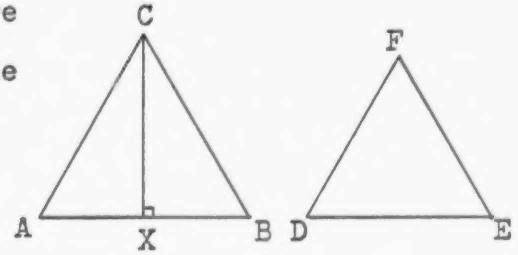
5. The areas of two similar triangles are 81 and 16. Find the ratio of any pair of corresponding sides.

6. The sides of a given triangle are 4, 5, and 7. Find the sides of a similar triangle whose area is nine times the area of the given triangle.

7. An equilateral triangle and a square have the same perimeter. Find the ratio of the area of the triangle to the area of the square.

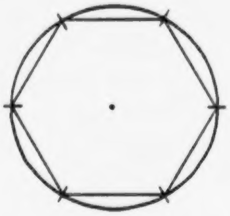
8. Given: $\triangle ABC$ and $\triangle DEF$ are equilateral $\triangle$s.
   $\overline{CX}$ is an altitude of $\triangle ABC$.
   $CX = DE$

   Prove: $\frac{A(\triangle ABC)}{A(\triangle DEF)} = \frac{4}{3}$



Regular Polygons

A lot of effort has been spent in the investigations of regular polygons and their relationship to circles. It can easily be seen that if a circle is divided into n nonintersecting congruent arcs, the polygon formed by drawing the chords of those arcs is a regular polygon. For the figure shown here, the given circle is divided into six congruent arcs, and the polygon formed by drawing all chords of these arcs can be proved to be a regular hexagon.

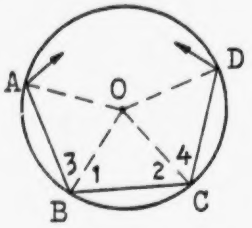


The next two theorems further emphasize this relationship by showing that for any given regular polygon a circle can be circumscribed about and a circle can be inscribed in that polygon.

Theorem 21-8: A circle can be circumscribed about any regular polygon.

Given: ABCD... is a regular polygon

Prove: There exists a circle that contains all the vertices of ABCD...



Argument

There exists a circle which contains any three noncollinear points.

① Let ⊙O contain points A, B, and C.

$$\text{②}\ \left.\begin{array}{r}\overline{OB} \cong \overline{OC} \xrightarrow{\text{③}} \angle 1 \cong \angle 2 \\ \text{④}\ \angle ABC \cong \angle BCD\end{array}\right\} \rightarrow \cdots \xrightarrow{\text{⑤}} \left.\begin{array}{l}\angle 3 \cong \angle 4 \\ \text{⑥}\ \overline{AB} \cong \overline{DC} \\ \text{⑦}\ \overline{OB} \cong \overline{OC}\end{array}\right\} \xrightarrow{\text{⑧}} \triangle AOB \cong \triangle DOC \xrightarrow{\text{⑨}}$$

$$\xrightarrow{\text{⑨}} \overline{AO} \cong \overline{DO} \xrightarrow{\text{⑩}} D \text{ lies on } \odot O$$

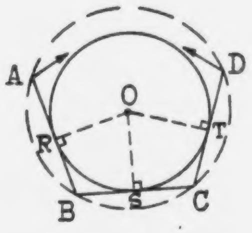
In like manner, each vertex of ABCD... can be shown to lie on ⊙O.

∴ There exists a circle that contains all the vertices of ABCD...

Theorem 21-9: A circle can be inscribed in any regular polygon.

Given: ABCD... is a regular polygon

Prove: There exists a circle tangent to each side of ABCD...



Argument

① Let O be the center of the circumscribed circle about ABCD...

② Then $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, ... are congruent chords of $\odot(O,A)$

③ Draw $\overline{OR} \perp \overline{AB}$, $\overline{OS} \perp \overline{BC}$, $\overline{OT} \perp \overline{CD}$, ...

④ Then $OR = OS = OT = \ldots$ since ≅ chords of a circle are equidistant from the center.

∴ The circle with center O and radius OR is tangent to each side of ABCD...

These two theorems are the basis for the following definitions about regular polygons.

The center of a regular polygon is the common center of its circumscribed and inscribed circles.

A radius of a regular polygon is a segment whose endpoints are the center and a vertex.

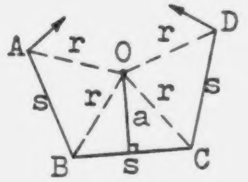
An apothem of a regular polygon is a segment from the center perpendicular to a side. The apothem will also mean the length of that segment.

A central angle of a regular polygon is an angle whose vertex is the center and whose sides contain consecutive vertices.

Theorem 21-10: The area of a regular polygon is half the product of the apothem and the perimeter.

Given: regular n-gon ABCD...
with center O,
perimeter p, and
apothem a

Prove: $A = \frac{1}{2}ap$



Argument

---

① Draw all radii from O.
② The n Δs thus formed are congruent by SSS.

$$\left.\begin{array}{r} \text{③ } A(ABCD\ldots) = n\cdot A(\Delta BOC) \\ \text{④ } A(\Delta BOC) = \frac{1}{2}as \end{array}\right\} \xrightarrow{\text{⑤}} \left.\begin{array}{r} A(ABCD\ldots) = n\cdot\frac{1}{2}as = \frac{1}{2}a(ns) \\ \text{⑥ } p = ns \end{array}\right\} \xrightarrow{\text{⑦}}$$

$$\xrightarrow{\text{⑦}} A(ABCD\ldots) = \frac{1}{2}ap$$

∴ The formula for area of a regular n-gon with apothem a and perimeter p, is $A = \frac{1}{2}ap$.

EXERCISE SET 21-4

Find the apothem, radius, and measure of a central angle of each of the following.

1. An equilateral triangle with side 8.
2. A square with side 10.
3. A regular hexagon with side 6.

Find the area of each of the following.

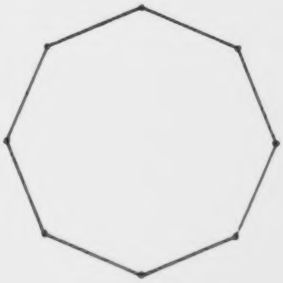
4. An equilateral triangle with side 6.
5. An equilateral triangle with apothem $2\sqrt{3}$.
6. An equilateral triangle with radius 8.
7. A regular hexagon with side 6.
8. A regular hexagon with apothem 6.
9. A regular hexagon with radius $\sqrt{3}$.
10. A regular pentagon with side 8 and apothem q.

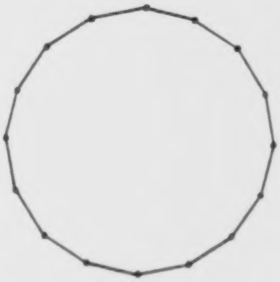
11. An equilateral triangle and a regular hexagon have equal areas. If the side of the triangle is 10, find the side of the hexagon.
12. Show that the area of a regular hexagon of side s is expressed by the formula $A = \frac{3}{2}\sqrt{3}\, s^2$.

Circumference of a Circle and π

The investigation of certain properties of a circle will test your powers of intuition as well as your powers of deduction. Consider the regular polygons shown below in which the number of sides is doubled and redoubled in the second and third figures.

4-gon


8-gon


16-gon

As the number of sides of the regular polygon increases, doesn't the shape of the regular polygon appear to approach the shape of a circle? We use this observation to discuss the circumference of a circle.

You have probably referred to the circumference as the "distance around" the circle. This is an acceptable descriptive phrase, but the language is a bit loose since we have not defined distance "along a curve". If $n = 1000$ in a regular n-gon, your eye would not be able to distinguish the 1000-gon from its circumscribed circle. The perimeter of the 1000-gon would be very nearly the circumference of the circle. However, the perimeter would always be less than the circumference since the length of any chord of a circle is always less than the "length" of its arc. This gives us a basis for defining the circumference of a circle.

The circumference of a circle is the limit of the perimeters of the inscribed regular n-gons as n increases without bound.

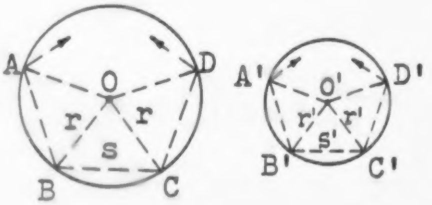
In other words, the circumference is the number that the perimeters approach but never exceed as the number of sides gets larger and larger. If you continue your study of mathematics, you will study the idea of limits more formally as it is a very vital concept. For now, the intuitive notion is sufficient for our purposes.

The ratio of the circumference to the diameter leads us to one of the most significant numbers in mathematics, the number π (pi). The next theorem establishes that this ratio is the same for all circles.

Theorem 21-11: The ratio of the circumference to the diameter is the same for all circles.

Given: ⊙O with circumference c
and diameter d
⊙O' with circumference c'
and diameter d'

Prove: $\frac{c}{d} = \frac{c'}{d'}$



Argument

① Inscribe a regular n-gon in each circle with sides s and s' as shown.
② Since each polygon has n sides, the perimeters are $p = ns$ and $p' = ns'$.
③ Draw radii of each regular n-gon r and r' as shown.

$$\left.\begin{array}{l} \text{④}\ m\angle BOC = \frac{360}{n} \\ \text{⑤}\ m\angle B'O'C' = \frac{360}{n} \end{array}\right\} \xrightarrow{\text{⑥}} \left.\begin{array}{r} m\angle BOC = m\angle B'O'C' \\ \text{⑦}\ \frac{OB}{O'B'} = \frac{OC}{O'C'} \end{array}\right\} \xrightarrow{\text{⑧}} \Delta BOC \sim \Delta B'O'C' \xrightarrow{\text{⑨}}$$

$$\xrightarrow{\text{⑨}} \frac{s}{r} = \frac{s'}{r'} \xrightarrow{\text{⑩}} \frac{ns}{r} = \frac{ns'}{r'} \xrightarrow{\text{⑪}} \frac{p}{r} = \frac{p'}{r'}$$

As n increases without bound, p approaches c as a limit,

so that ⑪ $\frac{p}{r} = \frac{p'}{r'} \xrightarrow{\text{⑫}} \frac{c}{r} = \frac{c'}{r'} \xrightarrow{\text{⑬}} \frac{c}{2r} = \frac{c'}{2r'} \xrightarrow{\text{⑭}} \frac{c}{d} = \frac{c'}{d'}$

∴ The ratio of circumference to diameter is the same for all circles.

Since the number $\frac{c}{d}$ is unique, it is given the special name π. The number π has been found to be irrational, and so π is the simplest name for it, just as $\sqrt{2}$ is the simplest name for the positive square root of 2. It is usually approximated as 3.14 or as $3\frac{1}{7}$, but if written as a decimal it becomes

3.141592653589...

This is an infinite nonrepeating decimal. With the use of advanced mathematics and computers, it has been carried out to over one hundred thousand places. The study of π has a rich place in the history of mathematics and applications of it are found in situations not directly related to circles.

The definition, $\frac{c}{d} = \pi$, leads directly to the familiar formula for finding the circumference when either the diameter or the radius is known.

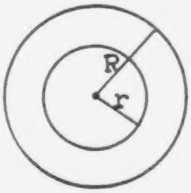
$$\frac{c}{d} = \pi \longrightarrow c = \pi d \longrightarrow c = 2\pi r$$

EXERCISE SET 21.5

1. 3.14 as a rational number to five decimal places is 3.14000.
   a) Write $3\frac{1}{7}$ as a rational number to five deciaml places.
   b) Is π less than both, greater than both, or between 3.14 and $3\frac{1}{7}$?
   c) Which number, 3.14 or $3\frac{1}{7}$, is closer to π?
2. Find the exact circumference of a circle whose radius is (a) 4 (b) 2.5.
3. Using $\pi \doteq 3.14$, approximate the circumference of a circle whose radius is
   a) 2 in.
   b) 15 cm.
   c) 7.5 ft.
   d) 4000 mi.
4. If a wheel has a 14 inch radius, approximately how far does it travel in one revolution?
5. Find the radius of a circle whose circumference is 10π.

6. The side of a square is 5. Find the circumference of its circumscribed circle and of its inscribed circle.
7. The side of an equilateral triangle is 6. Find the circumference of its circumscribed circle and of its inscribed circle.
8. The circumference of a circle is 20π. Find the side of a square inscribed in the circle.
9. The difference of the circumferences of two concentric circles is expressed by the formula

$$d = 2\pi R - 2\pi r \quad \text{or} \quad d = 2\pi(R - r)$$



   a) If $R = 2$ ft. and $r = 1$ ft., find the difference of the circumferences.
   b) If $R = 101$ ft. and $r = 100$ ft., find the difference of the circumferences.
   c) If $R = 1001$ ft. and $r = 1000$ ft., find the difference of the circumferences.

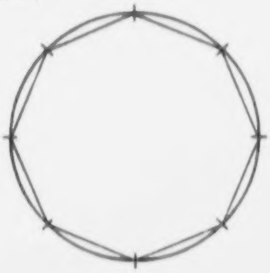
Area of a Circle

Of course, a circle doesn't have an area any more than a triangle or a polygon has. This leads us to a definition similar to those at the beginning of the chapter.

A circular region is the union of a circle and its interior.

So we will happily discuss the area of a circle knowing that we mean the area of a circular region. But we have defined area so far only as a number associated with a polygonal region. We will use the relationship between a circle and its inscribed regular polygons as discussed earlier to make the definition we need now.

Consider a regular n-gon inscribed in a circle. As n increases without bound, not only does the perimeter of the regular polygon approach the circumference of the circle as a limit, but the area of the regular polygon approaches the "area" (not yet defined) of the circle as a limit. This is the basis of the definition.



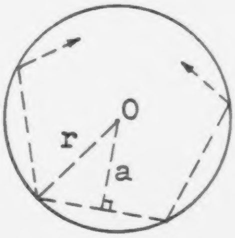
The area of a circle is the limit of the areas of the inscribed regular n-gons as n increases without bound.

This requires that the area of a circle also be a unique positive real number. We can now develop the familiar formula for area of a circle as a theorem.

Theorem 21-12: The area of a circle of radius r is πr^2.

Given: $\odot O$ with radius r
circumference c
and area A

Prove: $A = \pi r^2$



Argument

Inscribe a regular n-gon in $\odot O$ with apothem a, perimeter p, and area A_n.

$A_n = \frac{1}{2}ap$

Let n increase without bound
Then a approaches r as a limit
p approaches c as a limit
A_n approaches A as a limit

$$\text{Thus } A_n = \frac{1}{2}ap \longrightarrow \left.\begin{array}{r} A = \frac{1}{2}rc \\ c = 2\pi r \end{array}\right\} \longrightarrow A = \frac{1}{2}r(2\pi r) = \pi r^2$$

$\therefore$ The formula for area of a circle of radius r is $A = \pi r^2$.

Corollary 21-12a: The ratio of the areas of two circles is the square of the ratio of their radii.

EXERCISE SET 21.6

1. Find the exact area of a circle whose radius is

a) 5	c) $3\frac{1}{2}$
b) 200	d) $2\sqrt{3}$

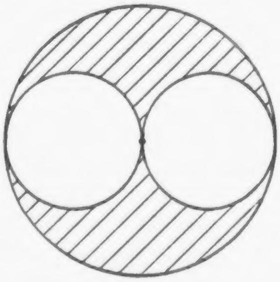
2. A regular hexagon has side 12. Find the area of its circumscribed circle and of its inscribed circle.
3. What is the radius of a circle whose area is 64π?

4. Find the area of the region bounded by two concentric circles with radii 9 and 6.

5. A 60 inch wire is bent into the shape of a square. Another 60 inch wire is bent into the shape of a circle. Using $\pi \doteq 3.14$, find the two areas to the nearest square inch. Which is greater and what is the difference?

6. A circular plaque with diameter 14 inches is mounted in the center of a square board with side 16 inches. What is the exposed area of the board?

7. Each of the smaller circles shown has a diameter that is half the diameter of the larger circle. If the diameter of the larger circle is 4, find the area of the shaded region.



8. Prove corollary 21-12a.

9. Two circles have radii 5 inches and 3 inches. What is the ratio of their areas?

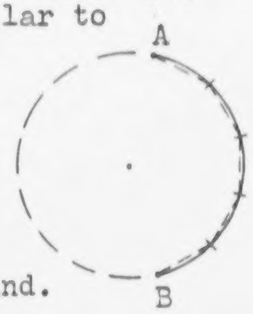
10. The ratio of the circumferences of two circles is 1:4. What is the ratio of their areas?

11. The ratio of the radii of two circles is 3:2. If the area of the larger circle is 36π, find the area of the smaller circle.

12. The area of an equilateral triangle is $9\sqrt{3}$. Find the ratio of the area of the circumscribed circle to the area of the inscribed circle.

Arcs, Sectors, and Segments of Circles

The measure of an arc has been defined as the degree measure. Another type of measure of an arc would be its length, which we will define in a manner similar to circumference.

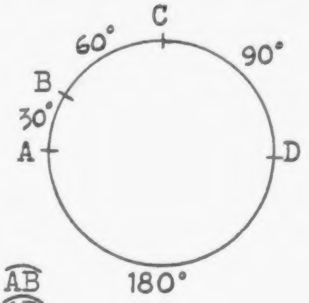


Let $\widehat{AB}$ be a given arc. Divide $\widehat{AB}$ into n congruent arcs and draw the chords of these arcs. Then the chords are also congruent.

The length of the arc $\widehat{AB}$ is the limit of the sum of the lengths of the n congruent chords as n increases without bound.

We will continue to use the word measure for degree measure, and length for this new type of measure.

For the figure shown, isn't it reasonable that the lengths of the arcs should have the same relationship to one another as the degree measures of those same arcs? That is,



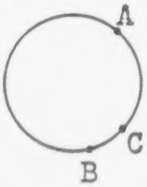
$$m\overset{\frown}{BC} = 2\cdot m\overset{\frown}{AB} \longrightarrow \text{length } \overset{\frown}{BC} = 2\cdot \text{length } \overset{\frown}{AB}$$
$$m\overset{\frown}{CD} = 3\cdot m\overset{\frown}{AB} \longrightarrow \text{length } \overset{\frown}{CD} = 3\cdot \text{length } \overset{\frown}{AB}$$

We will accept the generalization as a postulate.

Postulate 28: In the same circle or congruent circles, the lengths of arcs are proportional to their degree measures.

A circle itself may be considered as an arc whose degree measure is 360 and whose length is $2\pi r$. By postulate 28,

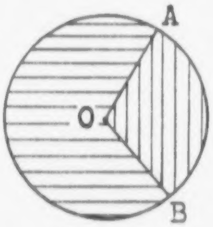
$$\frac{\text{length } \overset{\frown}{ACB}}{\text{length } \overset{\frown}{ABA}} = \frac{m\overset{\frown}{ACB}}{m\overset{\frown}{ABA}} \longrightarrow \frac{\text{length } \overset{\frown}{ACB}}{2\pi r} = \frac{m\overset{\frown}{ACB}}{360} \longrightarrow$$



$$\longrightarrow \text{length } \overset{\frown}{ACB} = \frac{m\overset{\frown}{ACB}}{360}\cdot 2\pi r \longrightarrow \text{length } \overset{\frown}{ACB} = \frac{m\overset{\frown}{ACB}}{180}\cdot \pi r$$

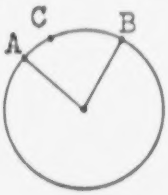
A sector of a circle is a region bounded by an arc of the circle and the two radii whose outer endpoints are the endpoints of the arc. The radius of the sector is the radius of the circle.

A central angle of a circle will determine two sectors, one in the interior of the angle and one in the exterior of the angle. The radii belong to each sector.



The area of a sector is defined in a manner similar to the area of a circle. For a given sector whose arc is $\overset{\frown}{ACB}$, the following proportion will be true.

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{m\overset{\frown}{ACB}}{360}$$



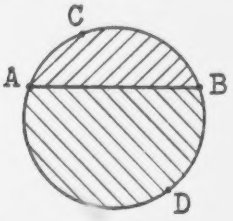
$$\therefore \text{ Area of sector with arc } \overset{\frown}{ACB} = \frac{m\overset{\frown}{ACB}}{360}\cdot \pi r^2$$

Since the length of $\widehat{ACB}$ is $\frac{m\widehat{ACB}}{180}\cdot\pi r$, an equivalent formula is

Area of sector $= \frac{1}{2}rl$, where l is the length of the arc.

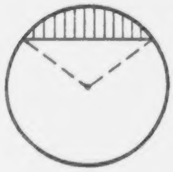
A segment of a circle is a region bounded by a chord and either arc of the chord.

Chord $\overline{AB}$ determines two segments of the circle, one bounded by $\widehat{ACB}$ and one bounded by $\widehat{ADB}$.



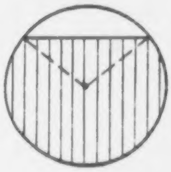
We really don't need a special formula for the area of a segment of a circle as we can use our present knowledge of area in the following manner.

$A(\text{segment}) = A(\text{sector}) - A(\text{triangle})$



or

$A(\text{segment}) = A(\text{sector}) + A(\text{triangle})$



EXERCISE SET 21.7

1. The radius of a circle is 12. Find the length of an arc that has degree measure

   a) 90 c) 135 e) 225

   b) 60 d) 36 f) 330

2. The radius of a circle is 6. Find the degree measure of an arc that has length

   a) π c) 12 e) 10π

   b) 3π d) 8π f) 6

3. The radius of a circle is 9. Find the area of a sector whose arc has measure

   a) 60 b) 90 c) 135 d) 210

4. The radius of a circle is 4. Find the area of a sector whose arc has length π.

5. An arc of a circle has degree measure 30 and length 2π. Find the radius of the circle.

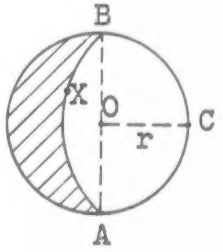
6. The arc of a sector has measure 144 and length 8π. Find the area of the sector.

7. A horse is tied to a 40 foot rope attached to the corner of a rectangular barn which measures 30 ft. by 50 ft. Using $\pi \doteq 3.14$, find the area to the nearest square foot of the region over which the horse can graze.

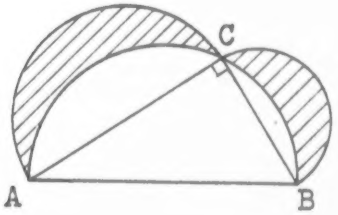
8. The radius of a circle is 8. Find the area of a segment whose arc has measure

   a) 60 b) 90 c) 120 d) 240

9. $\overline{AB}$ is a diameter of $\odot O$. $\overline{OC} \perp \overline{AB}$. c is the center of $\widehat{AXB}$. Prove that the shaded area is equal to r^2.



10. Each side of rt. $\triangle ABC$ is a diameter of a semicircle as shown. Prove that the area of the shaded region is equal to the area of $\triangle ABC$.



CHAPTER 22 - AREAS AND VOLUMES OF SPACE FIGURES

or WE'RE IN SOLID NOW

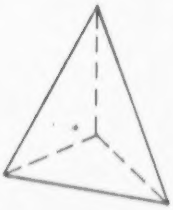
Polyhedrons and Area of a Prism

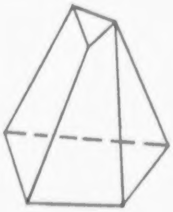
Space figures and some of their relations were considered in chapter 15. We will now look at some special space figures with respect to the ideas of area and volume. These are concepts that you are basically familiar with from your past studies. You have probably related the area and volume of a box to questions such as the following.

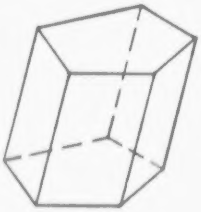
Area: How much paint does it take to cover the outside of it?
Volume: How much water does it take to fill it?

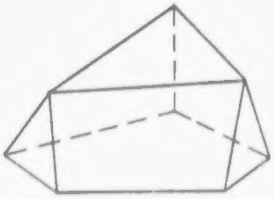
These notions are worthwhile in a physical sense, but of course, we will consider them in the abstract sense. However, our approach will be mostly informal with emphasis on the applications of the formulas.

The general name for closed space figures that are determined by intersecting planes is *polyhedron*. Some examples are illustrated below.









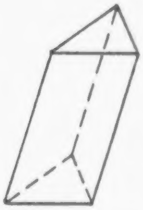
Each polygonal region is called a *face* of the polyhedron.
A vertex of a face is a *vertex* of the polyhedron.
A side of a face is an *edge* of the polyhedron.

The special polyhedrons that we will consider in this chapter are the prism and the pyramid. The other space figures we will work with are the cylinder, cone, and sphere.

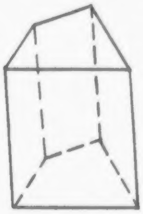
A *prism* is a polyhedron with the following conditions.

1. Two of the faces are bounded by congruent convex polygons lying in distinct parallel planes. These are the *bases*.
2. All other faces are bounded by parallelograms, each having one side in common with each base. These are the *lateral faces*. The intersection of two lateral faces is a *lateral edge*.

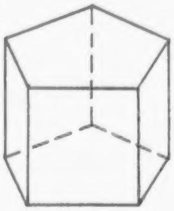
A prism is classified according to the polygon of its base.



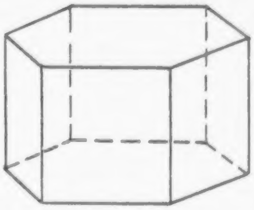
triangular prism



quadrantal prism



pentagonal prism



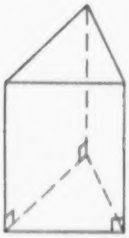
hexagonal prism

If the lateral edges are perpendicular to the base, the prism is a right prism.

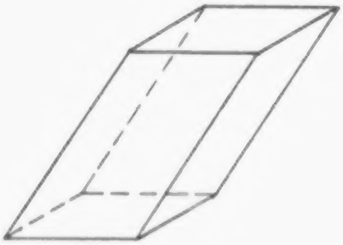
If the bases of a prism are bounded by parallelograms, the prism is a parallelepiped.

If all the faces of a prism are rectangles, the prism is a rectangular parallelepiped or right rectangular prism.

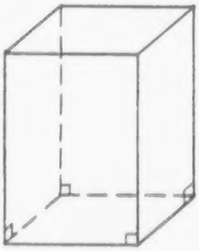
A rectangular parallelepiped all of whose faces are congruent is a cube.



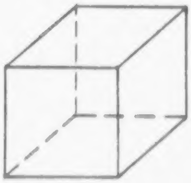
right prism



parallelepiped



right parallelepiped



cube

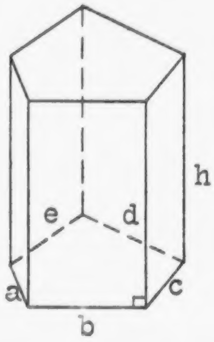
An altitude of a prism is a segment from any point in one base that is perpendicular to the plane of the other base.

The height of a prism is the length of an altitude.

The lateral area of a prism is the sum of the areas of its lateral faces.

The total area of a prism is the sum of the areas of all its faces. It is the lateral area plus the area of the two bases.

The formula for lateral area of a right prism is easily developed since all lateral faces are rectangles and each lateral edge is an altitude of the prism.



Lateral area $= ah + bh + ch + dh + eh$
$= (a + b + c + d + e)h$

But $a + b + c + d + e$ is the perimeter, p, of the base.

$\therefore$ The lateral area, L.A., of a right prism with base perimeter p and height h is expressed by the formula

$\text{L.A.} = ph$

The bases are congruent polygons with equal areas.

$\therefore$ The total area, T.A., of a right prism with base area B, base perimeter p, and height h is expressed by the formula

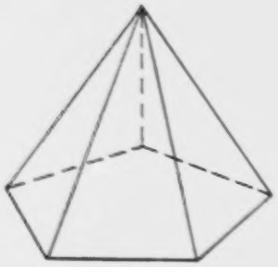
$\text{T.A.} = ph + 2B$

EXERCISE SET 22-1

1. The base of a right prism is a rectangle 20 by 30 and the height of the prism is 25. Find the lateral area and the total area.
2. The base of a right prism is an equilateral triangle with side 12 and the height of the prism is 20. Find Find the lateral area and the total area.
3. Find the total area of a cube with edge 8.
4. The height of a right pentagonal prism is 15 and the perimeter of the base is 22. Find the lateral area.
5. A block of wood is in the shape of a right rectangular prism with base dimensions 8 in. by 6 in. and a height of 15 in. It is cut into two pieces along a plane containing the diagonals of the bases. Find the total area of each piece.
6. The base of a right prism is a square. The height of the prism is 5 and the total area is 78. Find the side of the base.

Area of a Pyramid

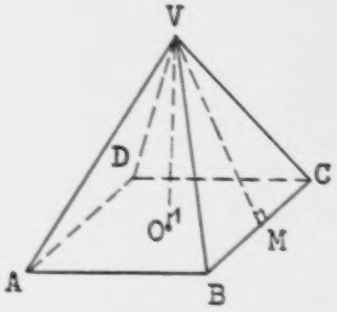
Given a polygonal region and a point not in the plane of the polygon, the union of the given polygonal region, which is the *base*, and all triangular regions determined by segments from the vertices of the base to the given point, is a *pyramid*.



The triangular regions are the *lateral faces* and the intersection of two lateral faces is *a lateral edge*. The common point of intersection of all lateral faces is *the vertex* of the pyramid. The *altitude* is the segment from the vertex perpendicular to the base, and the *height* is the length of the altitude.

A *regular pyramid* is a pyramid whose base is bounded by a regular polygon and whose lateral edges are all congruent.

It follows directly that the lateral faces are all isosceles triangles, and it can be shown that the altitude of the pyramid intersects the base at its center. For the regular pyramid shown, $\overline{VO}$ is the altitude of the pyramid and $\overline{VM}$ is the altitude of a lateral face. The length of $\overline{VM}$ is the *slant height* of the regular pyramid.



The *lateral area* of a pyramid is the sum of the areas of its lateral faces. The *total area* of a pyramid is the sum of its lateral area and the area of the base.

Formulas for these areas can be developed for regular pyramids.

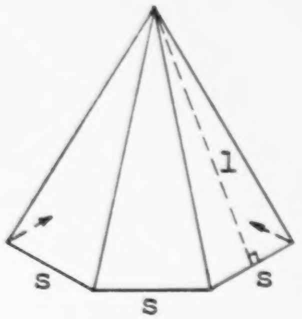
Let s be the length of an edge of the base and let l be the slant height of a regular pyramid.

The area of each lateral face is $\frac{1}{2}ls$.

If the number of lateral faces is n,

the lateral area is $n \cdot \frac{1}{2}ls = \frac{1}{2}(ns)l$.

But ns is the perimeter of the base, p.



$\therefore$ The lateral area, L.A., of a regular pyramid with base perimeter p and slant height l is expressed by the formula

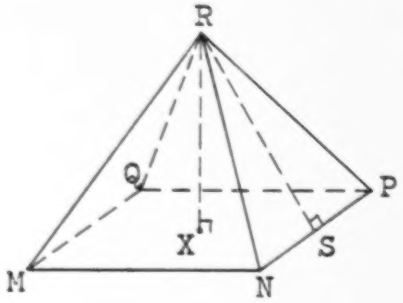
$$L.A. = \frac{1}{2}pl$$

The total area, T.A., of a regular pyramid with base area B, base perimeter p, and slant height l is expressed by the formula

$$T.A. = \frac{1}{2}pl + B$$

EXERCISE SET 22.2

1. For the square pyramid shown name each of the following.
    - a) The base
    - b) A lateral face
    - c) A lateral edge
    - d) The altitude
    - e) The height
    - f) The vertex
    - g) The slant height
    - h) An altitude of a lateral face

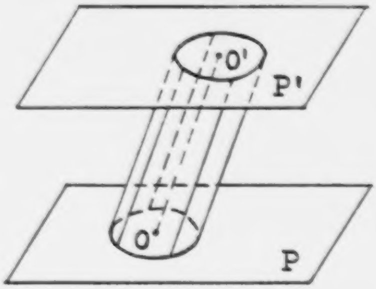


2. The base edge of a regular square pyramid is 8 and the slant height is 10. Find the lateral area and the total area.
3. Each face of a pyramid is bounded by an equilateral triangle. Find the total area if the length of each edge is a) 6 b) $\sqrt{3}$.

4. The height of a regular square pyramid is 3 and the slant height is 5. Find the length of a base edge and the lateral area.
5. The base edge of a regular hexagonal pyramid is 6 and the slant height is 8. Find the length of a lateral edge, the slant height, the lateral area, and the total area.
6. The measure of the dihedral angle formed by a lateral face and the base of a regular square pyramid is 60. If the height of the pyramid is 6, find the lateral area and the total area.

Areas of Circular Cylinders, Circular Cones, and Spheres

The sphere has already been defined as the set of all points in space at a given distance from a given point. The cylinder and cone are just as easy to "picture" but much more difficult to define.



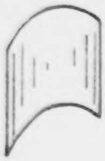
Let $\odot O$ and $\odot O'$ be congruent circles in distinct parallel planes P and P'. Let S_1 be the set of all points in the circular region of $\odot O$. Let S_2 be the set of all points in the circular region of $\odot O'$. Let S_3 be the *set* of all segments parallel to $\overline{OO'}$ whose endpoints are points of $\odot O$ and $\odot O'$.

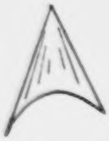
Then $S_1 \cup S_2 \cup S_3$ is a *circular cylinder*.

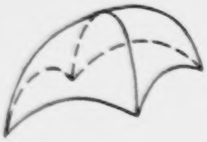
The circular regions S_1 and S_2 are the *bases* of the cylinder. S_3 is the *lateral surface* of the cylinder. Each segment of S_3 is an *element* of the cylinder. A segment from one base perpendicular to the plane of the other base is an *altitude* of the cylinder. The length of an altitude is the *height* of the cylinder.

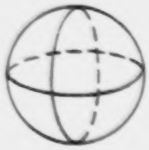
You would probably not question the meaning of the lateral area of a cylinder. But the word "area" has been defined as a number associated with a polygonal region of a plane. We then extended the meaning of area to a circle with the ideas of limits, but a circle still bounds a plane region. To justify a consideration of areas of curved surfaces, we will first accept the term *closed curved region* as undefined.

It can be described as a set of continuous noncoplanar points as illustrated below in which "closed" means it is not infinite in extent as a plane.









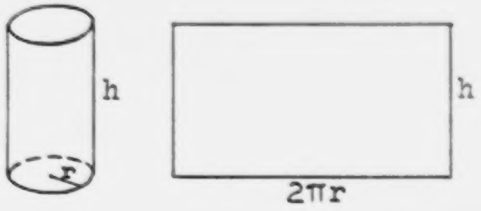
Postulate 29: For every closed curved region there is a unique positive real number equal to the area of a corresponding closed plane region.

The area of a closed curved region is the number assigned to it in postulate 29.

The lateral area of a circular cylinder is the area of its lateral surface.
The total area of a circular cylinder is the lateral area plus the area of the two bases.

A right circular cylinder is a circular cylinder whose elements are all perpendicular to the bases.

The lateral area of a right circular cylinder can be related to the area of a rectangle in the following manner.



Let r be the radius of the base and let h be the height of a right circular cylinder. If the cylinder could be cut along an element in a physical sense and laid out flat, the lateral surface would be a rectangular region with length equal to the circumference of the base, $2\pi r$. The width of the rectangle would be the height of the cylinder, h. The area of the rectangle would be the lateral area of the cylinder.

$A(\text{rect.}) = lw = (2\pi r)h$

$\therefore$ The lateral area, L.A., of a right circular cylinder with base radius r and height h is expressed by the formula

$\text{L.A.} = 2\pi rh$

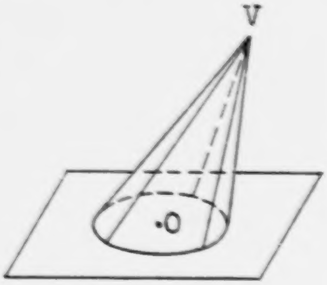
~

The total area, T.A., of a right circular cylinder with base radius r and height h is expressed by the formula

$$T.A. = 2\pi rh + 2\pi r^2$$

This development does not constitute a proof, but it does give us a correct formula.

An approach similar to that for the circular cylinder can be used for the definitions and formulas for the circular cone.



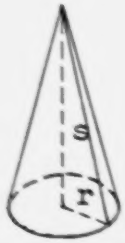
Let $\odot O$ be the boundary of a circular region and let V be a point not in the plane of $\odot O$. Let S_1 be the set of all points in the circular region of $\odot O$. Let S_2 be the set of all segments with one endpoint in $\odot O$ and the other endpoint in V.

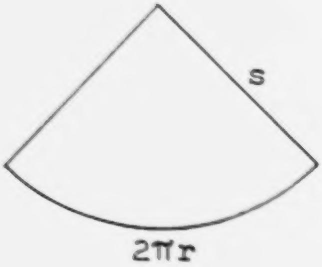
Then $S_1 \cup S_2$ is a circular cone.

The circular region is the base. S_2 is the lateral surface, the segments of S_2 are elements, and V is the vertex of the cone. The segment from V perpendicular to the plane of the base is the altitude, and the length of the altitude is the height. The lateral area is the area of the lateral surface and the total area is the lateral area plus the area of the base.

A right circular cone is a circular cone all of whose elements are congruent.

The altitude of a right circular cone intersects the base at its center. The length of an element of a right circular cone is its slant height.





Let r be the radius of the base and let s be the slant height of a right circular cone. The cutting of an element and flattening process would relate the lateral area to the area of a sector of a circle. The radius of the sector is the slant height of the cone, s. The length of the arc of the sector is the circumference of the base of the cone, $2\pi r$.

From the formula for area of a sector, we have

$$A(\text{sector}) = \frac{1}{2} rl = \frac{1}{2} s(2\pi r) = \pi rs$$

∴ The lateral area, L.A., of a right circular cone with base radius r and slant height s is expressed by the formula

$$\text{L.A.} = \pi rs$$

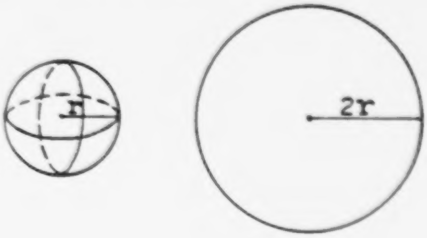
The total area, T.A., of a right circular cone with base radius r and slant height s is expressed by the formula

$$\text{T.A.} = \pi rs + \pi r^2$$

Any development of the formula for the surface area of a sphere is rather involved and will not be presented here. It can be shown that the area of a sphere is equal to the area of a circular region whose radius is twice the radius of the sphere.

$$A(\text{circle}) = \pi(2r)^2 = 4\pi r^2$$

$$A(\text{sphere}) = 4\pi r^2$$



EXERCISE SET 22.3

1. Find the lateral area and total area of a right circular cylinder with radius of base and height as follows.

   a) $r = 4$, $h = 7$ b) $r = 10$, $h = 1$

2. The radius of a right circular cylinder is 7 and the lateral area is 84π. Find the height.
3. The lateral area of a right circular cylinder is equal to the sum of the areas of the two bases. What is the relationship between the radius and the height?
4. Find the lateral area and total area of a right circular cone with radius of base and slant height as follows.

   a) $r = 5$, $s = 10$ b) $r = \frac{1}{2}$, $s = 2$

5. Find the lateral area and total area of a right circular cone with radius of base and height as follows.

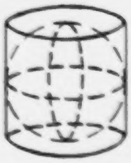
   a) $r = 5$, $h = 12$ b) $r = 3$, $h = 6$

6. The lateral area of a right circular cone is twice the area of its base. What is the relationship between the radius of the base and the slant height?

7. Find the area of a sphere if its radius is

   a) 3 b) 9 c) $\frac{1}{2}$

8. A cylinder is circumscribed about a sphere. Show that the area of the sphere is equal to the lateral area of the cylinder.



9. The diameter of the earth is approximately 8000 miles. The diameter of the moon is approximately 2000 miles. What is the approximate ratio of the area of the earth to the area of the moon?

Polyhedral Solids and Volume of a Prism

A polyhedron separates space into three sets; the polyhedron, its exterior, and its interior. We will accept the intuitive idea of exterior and interior rather than give formal definitions.

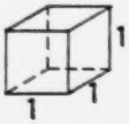
A polyhedral solid is the union of a polyhedron and its interior.

Our approach to volume of a polyhedral solid is similar to our approach to area of a polygonal region.

Postulate 30: To every polyhedral solid there corresponds a unique positive real number.

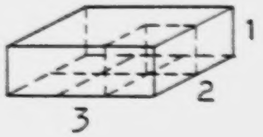
The volume of a polyhedral solid is the number assigned to it by postulate 30. The volume of solid S will be denoted V(S).

The unit of volume is defined as a cubic solid each of whose edges is 1 unit of length. It is 1 cubic unit.



We will relax our language with volume just as we did with area and refer, for example, to the volume of a prismatic solid as the volume of a prism.

We postulated the formula for area of a rectangle, $A = lw$. A reasonable followup is to consider the volume of a right rectangular prism. If the length and width of the base and height of the prism are all integers, we could use the counting procedure again to find the volume. For the figure shown, the base area is 6 square units and the volume is 6 cubic units. This suggests the following postulate.

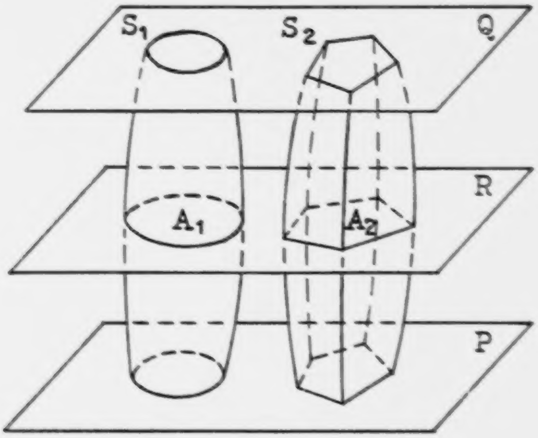


Postulate 30: The volume, V, of a right rectangular prism with base area B and height h is expressed by the formula $V = Bh$.

This is a small beginning, but it can be extended by the use of a very worthwhile principle first stated by an Italian mathematician, Bonaventura Cavalieri (1598-1647).

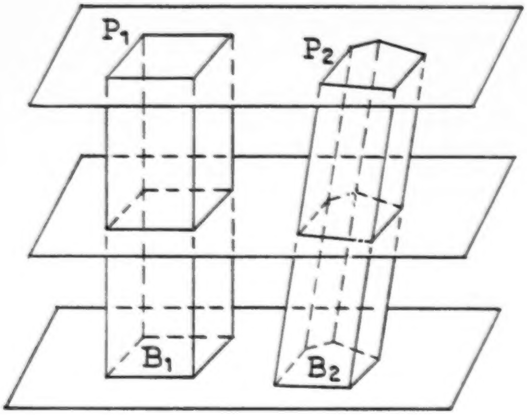
Cavalieri's Principle: If two solids are included between a pair of parallel planes, and if the two sections cut by them on any plane parallel to the including planes are equal in area, then the volumes of the solids are equal.

An illustration will help to clarify the statement.



Let P and Q be the including planes for solids S_1 and S_2. Let R be any plane parallel to P and Q and intersecting S_1 and S_2. If the cross sections in every plane R are equal in area, that is $A_1 = A_2$, then the volumes are equal, $V(S_1) = V(S_2)$.

It can be shown that any cross section of a prism parallel to the bases is a polygon congruent to the polygon of the bases. This is an important property that is necessary for the development of the next formula.



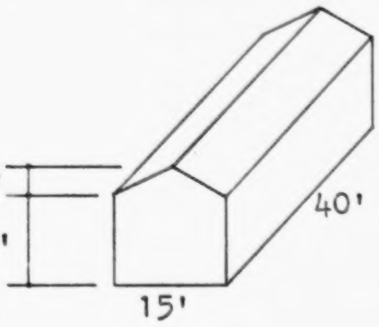
Let one solid be a right rectangular prism, P_1, and let the including planes be the planes containing the bases. Let P_2 be any other prism with bases in the including planes. If the base areas are B_1 and B_2, and $B_1 = B_2$, then every cross section of both prisms by planes parallel to the bases will be equal in area by the statement above the figure. By Cavalieri's Principle, the volumes of the prisms must be equal.

The volume of P_1 is base area times height and the prisms have the same base area and height.

$\therefore$ The volume, V, of any prism with base area B and height h is expressed by the formula $V = Bh$.

EXERCISE SET 22.4

1. The base area of a prism is 24 and the height is 5. Find the volume.
2. The dimensions of the base of a rectangular prism are 3 and 9 and the height is 10. Find the volume.
3. The base of a prism is an equilateral triangle with side 4. The height of the prism is 8. Find the volume.
4. The base of a prism is an isosceles right triangle with leg 6. The height of the prism is equal to the hypotenuse of the base. Find the volume.
5. Find the volume of a cube with edge 9.
6. The dimensions of a room are 18 ft. by 24 ft. by 9 ft. Find the volume of the room in cubic feet. Find the volume in cubic yards.
7. If each dimension of a rectangular prism is doubled, how is the volume changed?
8. If the numerical value of the volume of a cube equals the numerical value of its total area, find the length of an edge.
9. A shed has the shape and dimensions shown. Find its volume in cubic feet.



10. The volume of a rectangular prism is 160 and the height is 4. If the length of the base is three more than the width, find the length and width of the base.

Volume of a Pyramid

We can find the volume of a triangular pyramid by an interesting dissection of a triangular prism.

Let Fig. 1 below be a given triangular pyramid with base boundary ΔABC and vertex V.

Let Fig. 2 be a triangular prism with the same base and height.

Fig. 3 shows the dissection of the prism into three pyramids including the given pyramid. It also shows that the diagonal of each lateral face separates it into two congruent triangles.

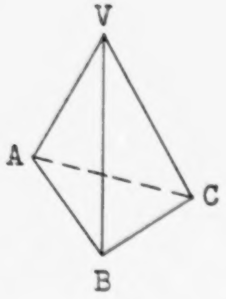


Fig. 1

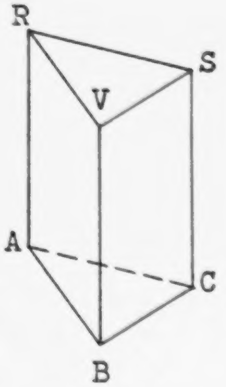


Fig. 2

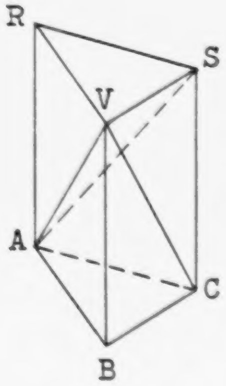
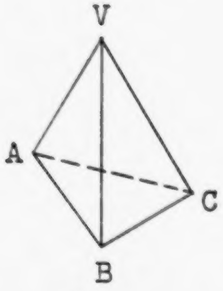
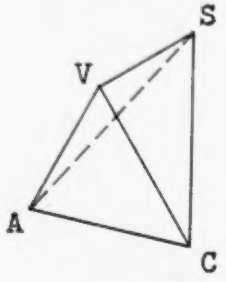


Fig. 3

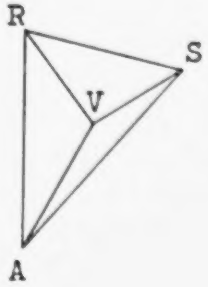
The pyramids of Fig. 3 are separated now for easier reference. Each is a triangular pyramid.



P_1



P_2



P_3

We will compare their volumes in pairs and we will name the base of each by the triangle that bounds it.

Let P_1 have base ΔVCB and vertex A.

Let P_2 have base ΔCVS and vertex A.

From Fig. 3, we see that $\Delta VCB \cong \Delta CVS$.

Then P_1 and P_2 have equal base areas in the same plane. Since they have the same vertex, they must also have the same height.

$\therefore V(P_1) = V(P_2)$

Let P_2 have base ΔSAC and vertex V.

Let P_3 have base ΔASR and vertex V.

From Fig. 3, we see that $\Delta SAC \cong \Delta ASR$.

Then P_2 and P_3 have equal base areas in the same plane. Since they have the same vertex, they must also have the same height.

$\therefore V(P_2) = V(P_3)$

All three pyramids have the same volume so each has a volume that is one third the volume of the prism.

The volume of the prism is expressed by $V = Bh$.

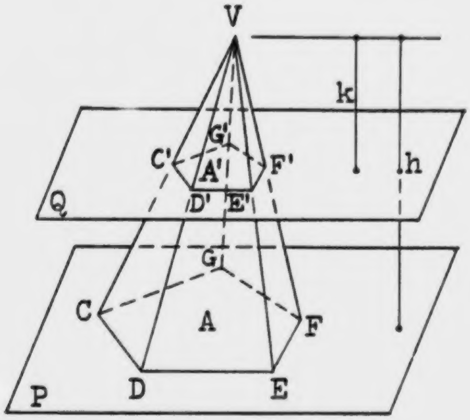
The given pyramid of Fig. 1 has the same base and height as the prism.

$\therefore$ The volume, V, of a triangular pyramid with base area B and height h is expressed by the formula

$$V = \frac{1}{3} Bh$$

We must now find a way to extend this formula to the volume of any pyramid.

In the figure below, P is the plane of the base of the pyramid. Q is any plane parallel to P intersecting the pyramid at a distance k from V. A is the area of the base and A' is the area of the cross section of Q with the pyramid. h is the height.



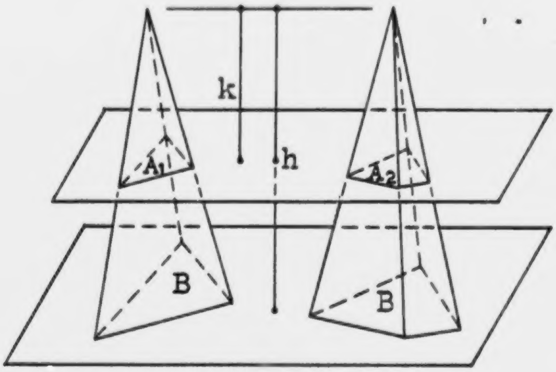
The conditions of the figure are very much like the conditions for a size transformation in a plane. We may consider it as a size transformation in space with center V and magnitude $\frac{k}{h}$ which maps CDEFG onto C'D'E'F'G'.

$$\therefore\ C'D'E'F'G' \sim CDEFG \quad \text{and} \quad \frac{C'D'}{CD} = \frac{k}{h}$$

Since the ratio of the areas of similar polygons is the square of the ratio of a pair of corresponding sides,

$$\frac{A'}{A} = \left(\frac{C'D'}{CD}\right)^2 = \left(\frac{k}{h}\right)^2 = \frac{k^2}{h^2}$$

In the next figure, the pyramids have equal base areas, B, and equal heights, h. A plane parallel to the plane of the bases intersects each pyramid as shown such that the areas of the cross sections are A_1 and A_2.



From the discussion on the preceding page, we have

$$\left.\begin{array}{l} \frac{A_1}{B} = \frac{k^2}{h^2} \\[1em] \frac{A_2}{B} = \frac{k^2}{h^2} \end{array}\right\} \longrightarrow A_1 = A_2$$

By Cavalieri's Principle, the volumes of the pyramids must be equal which gives the following statement.

If two pyramids have equal base areas and equal heights, then they have equal areas.

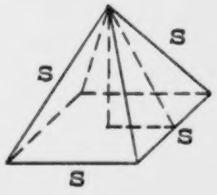
The formula for volume of a triangular pyramid with base area B and height h is $V = \frac{1}{3}Bh$.

∴ The volume, V, of any pyramid with base area B and height h is expressed by the formula

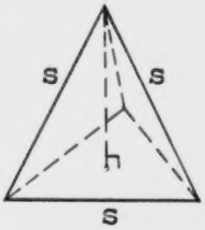
$$V = \frac{1}{3}Bh$$

EXERCISE SET 22.5

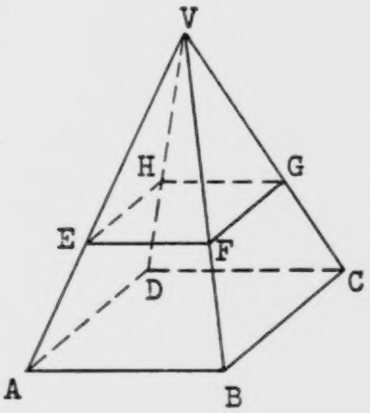
1. The base area of a pyramid is 20 and the height is 6. Find the volume.
2. The base of a square pyramid has side 8 and the height of the pyramid is 15. Find the volume.
3. The height of a regular hexagonal pyramid is 8 and the base edge is 6. Find the volume.
4. The base edge of a regular square pyramid is 8 and a lateral edge is 9. Find the volume.
5. A base edge of a regular square pyramid has side of length s. All the lateral faces are equilateral triangles with sides of length s. Write a formula for the volume of the pyramid in terms of s.



6. Every face of a pyramid is an equilateral triangle with side of length s. An altitude will intersect the opposite face at its center. Express the altitude in terms of s and write a formula for the volume of the pyramid in terms of s.

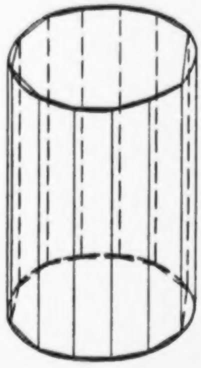


7. The triangular pyramid of exercise 6 is called a regular tetrahedron. Find the volume of a regular tetrahedron with edge 4.
8. In the figure shown, the plane of EFGH is parallel to the plane of ABCD. The solid having these polygons as bases and the trapezoids as lateral faces is called a frustrum of a pyramid. If the figure is a regular square pyramid with height 24, base edge 20, and $\frac{EF}{AB} = \frac{3}{4}$, find the volume of the frustrum.



Volume of a Circular Cylinder, Circular Cone, and Sphere

We approached our study of the circle in chapter 21 by considering the inscribed regular n-gons in which n increases indefinitely. Using a similar approach with solids, we can consider a prism that is inscribed in a circular cylinder and whose base boundary is a regular n-gon. As n increases indefinitely, we see that the volume of the prism approaches the volume of the cylinder as a limit. Therefore, we will accept the following statement.



The volume, V, of a circular cylinder with base radius r and height h is expressed by the formula

$$V = \pi r^2 h$$

The volume of a circular cone can likewise be considered as the limit of the volumes of the inscribed regular pyramids as the number of sides increases indefinitely.

The volume, V, of a circular cone of base radius r and height h is expressed by the formula

$$V = \frac{1}{3}\pi r^2 h$$

The formula for volume of a sphere can be justified with our present knowledge, but it is rather difficult. We will state it here without further discussion.

The volume, V, of a sphere with radius r is expressed by the formula

$$V = \frac{4}{3}\pi r^3$$

Each of these three formulas may be justified rigorously with the use of calculus which you may study in a later course.

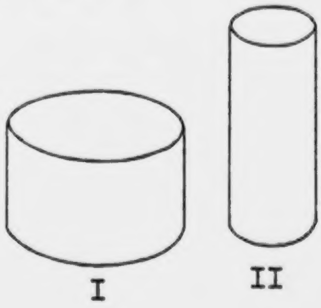
EXERCISE SET 22.6

1. Find the volume of a cylinder if the radius of its base and its height are as follows.

   a) $r = 5,\ h = 8$ c) $r = \frac{1}{2},\ h = 10$

   b) $r = 8,\ h = \frac{1}{2}$ d) $r = \sqrt{2},\ h = 3$

2. The diameter of cylinder I is twice the diameter of cylinder II. The height of cylinder II is twice the height of cylinder I. Compare their volumes.



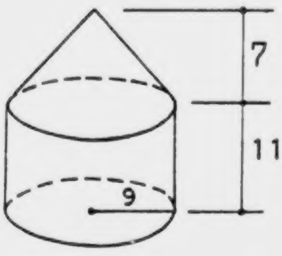
3. A metal pipe is 20 in. long. Its outside diameter is 3 in. and inside diameter $2\frac{1}{2}$ in. Using $\pi \doteq 3.14$, find the volume of the metal to the nearest tenth of a cubic inch.

4. Find the volume of a cone if the radius of its base and its height are as follows.

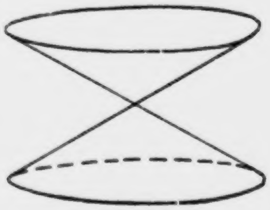
   a) $r = 6,\ h = 15$ c) $r = \frac{1}{2},\ h = 12$

   b) $r = 4,\ h = 10$ d) $r = \sqrt{3},\ h = \frac{1}{3}$

5. A storage building is in the shape of the figure. The radius of the cylindrical part is 9 ft. The total height of the building is 18 ft. and the height of the conical top is 7 ft. Find the volume of the building.

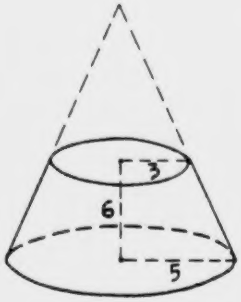


6. Two identical cones are related as shown. The radius of each base is 9 and the height of each cone is 5. Find the total volume. How does this volume compare with the volume of a single cone with the same base radius and height equal to the sum of the heights of the two cones?



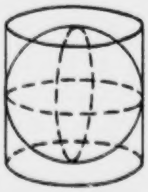
7. If the base radius and height of a cone are both doubled, how is the volume changed?

8. The lower portion of the figure shown is a frustrum of a cone. The radii of the two bases are 3 and 5 and the height of the frustrum is 6. Find the volume of the frustrum.



9. Find the volume of a sphere with radius
   a) 6 b) $\frac{1}{2}$ c) 10 d) $\sqrt{3}$

10. A cylinder is circumscribed about a sphere. Show that
    $\frac{V(\text{cylinder})}{V(\text{sphere})} = \frac{3}{2}$



11. If the numerical value of the volume of a sphere is equal to the numerical value of the area, find its radius.

12. The diameter of the earth is approximately 8000 miles. The diameter of the moon is approximately 2000 miles. What is the approximate ratio of the volume of the earth to the volume of the moon?